The Mathematics of Continuously Changing Objects

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Abstract

"The Mathematics of Continuously Changing Objects", which has been applied to business decisions, and can be applied to disease and commodities, has mostly been applied to human beings and their decision making priorities.

The paper determines an individual's progression through life by identifying him with a special matrix.Therefore it is not a statistical model.

Two competing goals are with him through life. One goal is determined by society such as money, power, fame, etc., by which he can compare himself with others, driven by his ego, represented by the origin.The other goal is to lead a fulfilling life driven by his instinctive drives, represented by the positive eigenvector of his matrix.

To determine the pace of his progression, a collection of matrices is used, all having the same eigenvectors as his own matrix.

Obstacles abound in life hindering his progression. They include marriage, employment, addiction, etc.. They are represented by matrix multiplication of the obstructing matrix with his own matrix.

The Perron-Frobenius Theorem plays a central role.

Introduction

Aristotle considered motion and rest to be mutually exclusive. If an object occupied a particular position at a particular time, it was at rest. This concept led to Zeno's third paradox: how could an object ever be moving since, at any instant of time it occupies a given position and is therefore at rest?

We moved on to another paradigm with Newton's mechanics. We investigated what made objects move and considered objects moving at a constant velocity to be essentially at rest. Calculus came in as the study of continuous changes of fixed objects and showed how various paths of these objects can be determined from certain requirements, eg. differential equations, calculus of variations.

Today we smash objects to find new objects that we identify by their paths in a bubble chamber, objects that were predicted by our theories. At present, much of our mathematics is focused on pathways of fixed objects over fixed geometric surfaces.

Objects in the real world however are not fixed. They change continuously, including ourselves. This paper offers a mathematics that examines these changing objects using mathematical techniques to obtain solutions to some of our individual real life problems.

Change

We live in a universe that is constantly changing. No object remains unchanged. The amount of time necessary to detect a change can be anywhere from a femtosecond to billions of light years.

Even before we can discuss the change of an object, we must define the state of the object before it changes. Only if we know that the state of the object now is different from what it was before can there be any meaningful discussion. We begin with the following definition.

Definition. Given an entity. A framework is an ordered set of parts or dimensions of the entity. Each part is assigned a positive number that represents an amount of a common substance U accumulated over a given time interval.

1. The entity is a person. The framework is the P(physical),I(intellectual), E(emotional), and S(spiritual) parts of himself. U is time. The numbers are the time spent in each coordinate expressed in hours over a given day.

2. The entity is a company. The framework is n of its stores.U is money. The numbers are the amount of money made in each store expressed in dollars over a given business cycle

3. The entity is a particular disease. The framework is the countries of the world. U is people. The numbers are the number of people who have that disease in each country, counted over a given year.

4. The entity is a conglomerate. The framework is the n countries who are members. U is the commodity “corn”. The numbers are the amount of corn harvested in each country expressed in bushels over a given harvest season.

In abstract form, a framework of n dimensions leads to an n dimensional vector of positive numbers. If we look at two different time periods and find that the number in one of its dimensions has remained unchanged, it does not mean there was no change in that dimension. It simply means that the difference was too small to measure. Therefore the real positive numbers are the most appropriate numbers to use in general.

The number 0 has no place in a changing universe. It represents no change, and as such cannot be use as a legitimate number in a framework. It is a boundary of the positive numbers. When it is crossed, it leads to an absurd world where change becomes undefined as we shall see. Thus we will be investigating the set of n dimensional real valued vectors with positive entries.

Instead of considering a vector as having a length and direction as is commonly done in science, we regard a vector as a distribution by dividing by its sum of coordinates. When we do that we specify the vector P by [P].

Definition. The state of a vector V is defined by its distribution [V].

Definition. Two vectors P,Q are in the same state if [P]=[Q]. Notice that [P]=[kP] for k>o.

Any of our vectors V serve a dual purpose. If we add its coordinates, we get a number that represents an amount of U and each coordinate a part of that amount. Since [V]=[kV], the state of the vector does not depend on any particular amount. It allows us to take any amount we choose and the coordinates of [V] will be percents of that amount. What is essentially the same thing, the state of a vector is a probability distribution on the numbers 1 to n, each number representing a coordinate.

Definition. We say that P and Q are equivalent, in symbols $[P]\sim[Q]$ if $|p_i/\Sigma p_i - q_i/\sum q_i|$

$< e, i=1,2,\ldots n$, for a fixed value of e.

The number e represents the limit of what we consider close enough to being in the same state, since the difference will have no significance in our investigation.

For the state of an individual using the P,I,E,S framework, any change that is less than a minute in time spent in each coordinate per day is considered essentially the same. For the time interval of one day, there are 1440 minutes. Now $1440 \times .0007 = 1.008$ minutes. It is reasonable to let $e = .0007$.

Order

In order to formulate a mathematics that deals with real life situations, it is necessary to begin with our everyday lives. These problems first originated within ourselves as human beings. Mathematics represents the abstraction of these problems, following and developed from basic structures of human thought patterns.

Let us examine some point in an individual's life where a concept first begins to blossom.

Big sister, age 4 or 5, has been sent in by Mother to comfort baby brother who has awoken rather tearfully from his afternoon nap. She has been drinking milk, and brings the glass in with her. Baby brother, suitably calmed by her presence, begins to play with some toys in his crib. Sister, seeing no further need for her presence, scampers back to her TV show in another room. Before she goes, however, she places the now empty glass on a dresser near the crib. The ever alert baby stands up, stretches for the glass just within his reach, and in the natural, clumsy way of a baby, knocks it off the dresser. It shatters into many pieces with a resounding crash.

Now baby is seeing something happen for the first time in his young life, and it's a terrifying experience: something gets completely demolished right before his eyes. Never had it occurred to him that such a thing was possible. Well, baby begins screaming at the top of his lungs, bringing Mother rushing in faster than she was previously moving when she heard the initial crash of the glass. She picks up baby and soothes him until he quiets down. But within that small brain of his , he has been made aware of something previously unknown. He saw that glass shatters and as he develops, his mind will be able to abstract from that particular experience the notion of "shattering". He will begin to suspect that other glasses might be shattered when dropped, and that objects other than glass may have the same property. In fact, as he grows older, the notion of which things shatter and which do not will become increasingly more important to him, and in many cases, form a basic part of his investigation of the world around him. He will discover that objects have different degrees of '"shatterability". Unlike the glass, which had to be dropped to shatter, the robin's egg his sister let him hold shattered by his squeezing it too hard. On the other hand, his rubber ball, instead of shattering when dropped, bounced back in as excellent condition as before it was dropped—that is, until his father ran over it with the family car. Then it was squooshed beyond recognition. So not only do all objects seem capable of being shattered, but different objects require different exertions before this happens.

What is happening is that the baby, now a child, has developed within himself a kind of scale, one that measures the degree of shatterability of objects. Eventually and Inevitably he is one day going to look at his body, recognize that it too can be shattered, and with that realization, the notion of death will follow quickly on its heels.

There are many similar, very basic concepts which we carry along to adulthood, and which we count on in each other for a basis of communication and understanding; we are at home within our species. So it is that in every situation, we take into our mentality many such varying aspects of things, some of them nicely equipped with their own numerical scale.

Weight is one such property; our weight and the weight of objects is assigned a number obtained by a weight scale devised by the use of a spring. Likewise with height and a scale which can easily be devised by putting marks on a rod. Age, another varying property, gets its numbers from a different kind of scale called a calendar. Economic worth of an object has numbers associated with it called a price. Any time there is a varying property, some sort of scale is always lurking in the background. Hotness and coldness of things has a scale called a thermometer, which involves a column of mercury. Intelligence is measured with limited success by use of a scale called an I.Q. test.

In the past century, when we are asked to give references for a job, what does the employer do? He sends out one of several types of printed forms to the individuals he"s been referred to, who the perspective employee claims knows him well, for informative and beneficial judgement. These forms generally have the following instructions: On a scale from 1 to 10, (10 being the best.), rate this person with respect to initiative, perseverance, honesty, dependability, emotional stability, ability to fit well in a group, etc.

The reference is asked to work within this scale. He evaluates the individual with a series of numbers and the employer can skim over the responses and construct from the number values a sketched out opinion of this person.

Given a framework. , We can think of each coordinate as a scale measuring the degree of satisfaction in that coordinate, represented by the number x_i. Assuming that the larger the number, the more satisfaction there is, we can imagine an ideal vector $G(g_1 g_2 \ldots g_n)$ representing the most satisfaction that one could possibly expect in the situation at hand . Such a vector represents the goal . Once G is given, a partial order is imposed on the set of positive vectors(a poset).

, Let X be the set of all points $X=(x_1x_2\ldots x_n)$ such that $x_i<g_i$ $i=1,2,\ldots n$, x_i in R.

Definition The points will be said to be in standard form, if we assign new coordinates to each point in X and to G as follows:

X will be assigned coordinates g_i-x_i and $G(0,0,\ldots0)$

Thus in standard form, the goal G will be the origin and X will be a set of n-tuples of real numbers with positive entries.

Definition. We say $X\leq Y$ if $x_i\leq y_i$, , $X<Y$ if $X\leq Y$, $X\neq Y$, and $X<<Y$, if $x_i<y_i$, $i=1,2,\ldots n$.

Once we have a poset, if $X<Y$ and we have to choose between them we will certainly choose X, since it is better than Y. However, if X is noncomparable to Y, then there is no better and yet we must make a choice. We have, in the past, found many ways of making one... We can leave it up to fate by flipping a coin, rely on tradition if applicable, turn the vectors into a simple order by taking the average of their coordinates, the distance from the vectors to the origin, allowing one of the dimensions to be the deciding factor, etc. In any case, we make our choice.

Activities

From the earliest age ,the toddler engages in activities that improve him. He goes from crawling, to walking, to running; developes skills; eating with utensils, drinking from a cup.

He starts playing games with other children. He sees that if someone keeps beating him in some game, then it will keep happening if he does not improve. All these games involve a goal, which defines an order, and which in all cases can be referred to as the origin.

These are the activities of childhood, which, as the person grows to adulthood, the activities develop and the goals shift from winning games and coming out favorably in childish contests, to goals of acquiring money, power, fame, etc. in adult life. We are judged by the goals that we strive toward, the pace we are moving, the small wins and losses we get along the way. We develop an ego, a self importance in the game of life, a competition with each other.

Thus we expect two basic properties with any of our activities:

1) it preserves order between any two vectors.

2) there is a vector X that gets improved.

Order preserving is a comparison between two vectors before and after each engages in the same activitiy.

Improvement is a comparison of a given vector before and after engaging in an activity.

Mathematically, one of the simplest activities that one can perform on a set of n-dimensional vectors is an nxn matrix.

Expressed in mathematical terms, if A is an nxn matrix, then:

Order preserving: For all X,Y, if $X<Y$ then $AX<AY$

Improvement: There exists an X such that $AX<X$.

From these humble beginnings, it is remarkable how far we can get to replicating the human experience, including symmetry breaking in quantum mechanics to black holes in general relativity. One wonders if ,in spite of all the experiments we are performing, our interpretation of them falls back to how the human mind is able to process information rather than discovering an external reality of our natural world, independent of our mind. It never ceases to amaze us the close connection between science and mathematics. But mathematics comes solely from the mind. It is remarkable only if we think they are separate.

Definition. Any matrix that is order preserving and improves at least one vector, will be called an activity matrix.

Theorem1. A is order preserving if and only if A has all nonnegative entries.

The proof is trivial.

Theorem2. Let A be an nxn matrix with nonnegative entries and suppose $AX<X$ for some X in X. Then

(i) *the diagonal elements a_{ii} of A are such that $0 \le a_{ii} < 1$, $i=1,2,\ldots,n$.*

(ii) *the co-factors U_{ij} of $I-A$, I the identity matrix, are all non-negative and $0 < U_{ii} < 1$.*

(iii) $0 < \chi(1) < 1$ *where $\chi(x)$ is the characteristic polynomial of A, i.e. $\chi(x) = |xI - A|$.*

Proof. The conditions of the theorem imply that $I-A$ is a non-singular M matrix (see Burman, Plemmons [4], p.136, I_{28}). Thus $1-a_{ii} > 0$, $U_{ii} > 0$ and $\chi(1) > 0$ since all the principal minors of $I-A$ are positive.

Furthermore $(I-A)^{-1} > 0$ implying that $U_{ij} \ge 0$. $U_{ii} < 1$ and $\chi(1) < 1$ are trivial consequences. This proves the theorem. ■

In what follows, we shall have occasion to use the expression $1-\chi(1)$ which will be denoted by p. Thus (iii) is equivalent to

(iii') $0 < p < 1$

Definition. *The set of all vectors $X \in \mathbf{X}$ such that $AX << X$ will be called the improvement region of A and denoted by $\mathbf{I}_A$. Let*

$$V_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ \cdot \\ \cdot \\ U_{in} \end{bmatrix}, \qquad i = 1,2,\ldots,n.$$

From Theorem 2, the V_i have all non-negative entries. The next theorem shows that $\{V_1, V_2, \ldots, V_n\}$ forms the boundary vectors of a conical region whose interior is the improvement region $\mathbf{I}_A$.

Theorem 3. *$X \in \mathbf{I}_A$ if and only if $X = \sum_{i=1}^{n} c_i V_i$ where $c_i > 0$, $i = 1,2,\ldots,n$.*

Proof. Since the column vectors of Adj$(I-A)$ are the V_i's we have

$$(I-A)\mathrm{Adj}(I-A) = \chi(1)I$$

and it follows that $AV_i = V_i - \chi(1)I_i$ where I_i is the i'th column of the identity matrix.

Since the V_i are independent they form a basis for R^n. Thus, for any $X \in R^n$ we can write

$$X = \sum_{i=1}^{n} c_i V_i$$

and

$$AX = \sum_{i=1}^{n} c_i A V_i = \sum_{i=1}^{n} c_i (V_i - \chi(1) I_i)$$

$$= X - \chi(1) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Hence $X \in \mathrm{I}_A$ if and only if $c_i > 0,\ i = 1, 2, \ldots, n$. This proves the theorem. ■

Corollary 3.1. *If* $0 < p < 1$, *then* $\mathrm{I}_A \neq \phi$.

Proof. $0 < p < 1$ is equivalent to $0 < \chi(1) < 1$. Thus it is only necessary to observe that for any

$$X = \sum_{i=1}^{n} c_i V_i, \qquad c_i > 0, \quad i = 1, 2, \ldots, n$$

then

$$AX = X - \chi(1) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} << X$$

This result coupled with Theorem 2 yields

Corollary 3.2. $\mathrm{I}_A \neq \phi$ *if and only if* $0 < p < 1$.

Theorem 4. *Let* A *be an* $n \times n$ *matrix with non-negative entries such that* $\mathrm{I}_A \neq \phi$. *For any* $X \in \mathrm{I}_A$,

$$A^n X < pX.$$

Proof. We will show that $A^n V_i < p V_i$ for $i = 1, 2, \ldots, n$ where the V_i's are the vectors bounding I_A. Since any $X \in \mathrm{I}_A$ is such that

$$X = \sum_{i=1}^{n} c_i V_i, \qquad c_1, c_2, \ldots, c_n > 0$$

the theorem will follow.

From the proof of Theorem 3 we saw that

$$AV_i = V_i - \chi(1) I_i .$$

Claim 1. $U_{ii} - \chi(1) < [1-\chi(1)]U_{ii} = pU_{ii}$

Proof. $[1-\chi(1)]U_{ii} - [U_{ii} - \chi(1)] = \chi(1)[1-U_{ii}] > 0$ since $\chi(1) > 0$ and $U_{ii} < 1$.

But $U_{ii} - \chi(1)$ is the i'th entry of AV_i. Thus that entry has already been diminished by the factor p after only one application of A. Now

$$A^n V_i = V_i - \chi(1)[I_i + AI_i + \ldots + A^{n-1}I_i] = [1-\chi(1)]V_i - \chi(1)[I_i \in AI_i + \ldots. + A^{n-1}I_i - V_i]$$

Claim 2. $I_i + AI_i + \ldots + A^{n-1}I_i - V_i \geq 0$. Now

$$AI_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ a_{ni} \end{bmatrix}$$

$$A^2 I_i = \begin{bmatrix} \sum_{j_1=1}^{n} a_{ij_1} a_{j_1 i} \\ \cdot \\ \cdot \\ \sum_{j_1=1}^{n} a_{nj_1} a_{j_1 i} \end{bmatrix}$$

$\cdot$
$\cdot$
$\cdot$

$$A^k I_i = \begin{bmatrix} \sum_{j_{k-1}} \sum_{j_{k-2}} \cdots \sum_{j_1} a_{ij_{k-1}} a_{j_{k-1} j_{k-2}} \cdots a_{j_2 j_1} a_{j_1 i} \\ \cdot \\ \cdot \\ \sum_{j_{k-1}} \sum_{j_{k-2}} \cdots \sum_{j_1} a_{nj_{k-1}} a_{j_{k-1}} a_{j_{k-2}} \cdots a_{j_2 j_1} a_{j_1 i} \end{bmatrix}$$

Notice that each entry of $I_i + AI_i + \ldots + A^{n-1}I_i$ is made up of sums of products of entries of A. We shall refer to a product of k entries of A as a k factor term. Thus, AI_i consists of a sum of 1 factor, $A^2 I_i$, a sum of 2 factors, . . . , $A^{n-1}I_i$, a sum of $(n-1)$ factors. Also

$$V_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ \cdot \\ \cdot \\ U_{in} \end{bmatrix}$$

Consider a given entry U_{ij} where $i \neq j$. Since U_{ij} is the ij'th co-factor of $I-A$, if we expand it, the resulting expression will consist of sums and differences of k factors for $k=1,2,\ldots,n-1$. Subtracting from $(I_i)+A(I_i)+\ldots+A^{n-1}(I_i)$, one can note that every k factor term for $k<n-1$ that ends up as a negative term can be canceled with the identical positive term of $(I_i)+A(I_i)+\ldots+A^{n-1}(I_i)$. As for the $(n-1)$ factor terms, either they can also be canceled in the same way, or they are of the form

$$-\prod_{j=1}^{n-1} a_{ij}a_{kk}$$

for some $k=1,2,\ldots,n$ in which there is a positive term $\prod_{j=1}^{n-1} a_{ij}$ also from U_{ij}. Combining these two terms, we get

$$\prod_{j=1}^{n-2} a_{ij}(1-a_{kk}) \geq 0$$

since $a_{kk}<1$. Thus,

$$I_i + AI_i + \ldots + A^{n-1}I_i - V_i > 0.$$

This proves the claim. Since

$$A^n V_i = pV_i - \chi(1)[I_i + AI_i + \ldots + A^{n-1}I_i - V_i],$$

it follows that

$$A^n V_i < pV_i$$

This proves the theorem. ■

Corollary 4.1. *Given the conditions of Theorem 4,*

$$A^{kn}X < p^kX$$

Proof. Since A is order-preserving

$$A^{2n}X = A^n(A^nX) < A^n(pX) = pA^nX < p^2X.$$

Thus, $A^{kn}X < p^k X$.

From Theorem 4, we see that p represents a guaranteed amount of improvement for any vector in I_A after n applications of A. For that reason p will be referred to as the *rate of improvement* of A. Note that any particular vector $X \in I_A$ may improve at a faster rate than p, but none can improve more slowly. p sets the minimal amount of improvement for any factor of any vector in I_A after n applications of A.

Definition. *A square non-negative matrix A is said to be primitive if there exists a positive integer k such that $A^k >> 0$.*

Theorem 5. *If A is a primitive matrix such that $I_A \neq \phi$, then*

(i) $\lambda < 1$ *where λ is the Perron Frobenius eigenvalue*

(ii) *for any $X \in \mathbf{X}$,* $\lim_{N \to \infty} \frac{x_i^{(N)}}{x_i^{(N-1)}} = \lambda$, *where $x_i^{(N)}$ is the i'th row of the column vector* $A^N X$.

(iii) *if $X \in I_A$, then*

$\min_{1 \le i \le n} \frac{x_i^{(N)}}{x_i^{(N-1)}}$ *monotonically increases to* λ

and

$\max_{1 \le i \le n} \frac{x_i^{(N)}}{x_i^{(N-1)}}$ *monotonically decreases to* λ

as $N \to \infty$.

Proof. Since $I - A$ is a non-singular M matrix (i) is an immediate consequence of the Perron Frobenius theorem (see [14], pp. 3-4). It also follows from that theorem that there is an eigenvector of λ with all positive entries. We shall denote such an eigenvector by Γ_n. Assume at first that all the eigenvalues of A are distinct. Then a set of corresponding eigenvectors Γ_i form a basis of R^n. Thus any $X \in R^n$ can be written as $X = \sum_{i=1}^{n} c_i \Gamma_i$. Since $\lambda = \lambda_n$ is a real number larger than the modulus of any other eigenvalue,

$$\lim_{N \to \infty} \frac{1}{\lambda^N} A^N X = \begin{cases} c_n \Gamma_n & \text{if } c_n \neq 0 \\ 0 & \text{if } c_n = 0 \end{cases}$$

Claim 1. If $X \in \mathbf{X}$ then $c_n \neq 0$

Proof. We can find an $\varepsilon > 0$ so that $\varepsilon\Gamma_n < X$ for any $X \in \mathbf{X}$. But $1/\lambda^N A^N(\varepsilon\Gamma_n) \leq 1/\lambda^N A^N X$. The left-hand side tends to $\varepsilon\Gamma_n$ as $N \to \infty$. Hence $\lim_{n \to \infty} 1/\lambda^N A^N X \geq \varepsilon\Gamma_n > 0$. Therefore $c_n \neq 0$. In fact $c_n > 0$. Let $\Gamma_j = \begin{bmatrix} \delta_{j1} \\ \cdot \\ \cdot \\ \delta_{jn} \end{bmatrix}$. Then

$$\frac{x_i^{(N)}}{x_i^{(N-1)}} = \frac{\sum_{j=1}^{n} \lambda_j^N c_j \delta_{ji}}{\sum_{j=1}^{n} \lambda_j^{N-1} c_j \delta_{ji}} = \frac{\lambda_n^N}{\lambda_n^{N-1}} \frac{\sum_{j=1}^{N} \left[\frac{\lambda_j}{\lambda_n}\right]^N c_j \delta_{ji}}{\sum_{j=1}^{n} \left[\frac{\lambda_j}{\lambda_n}\right]^{N-1} c_j \delta_{ji}} \to \lambda_n = \lambda \text{ as } N \to \infty.$$

If not all the eigenvalues are distinct then we can vary the matrix so that they are, and by continuity we obtain the same result.

Lemma 5.1. *Suppose* $X \in \mathbf{I}_A$. *If*

$$AX = A\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 x_1 \\ p_2 x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix}$$

and

$$A^2X = \begin{bmatrix} q_1 p_1 x_1 \\ q_2 p_2 x_2 \\ \cdot \\ \cdot \\ q_n p_n x_n \end{bmatrix},$$

then the q*'s lie between the* p*'s.*

Proof.

Suppose $p^* = \max_{1 \leq i \leq n} \{p_i\}$. then $\begin{bmatrix} p_1 x_1 \\ p_2 x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix} < p^* \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$.

Since A is order-preserving

$$A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n\,x_n \end{bmatrix} \leq p^* A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n\,x_n \end{bmatrix}. \text{ Thus } \begin{bmatrix} q_1p_1x_1 \\ q_2p_2x_2 \\ \cdot \\ \cdot \\ q_n\,p_n\,x_n \end{bmatrix} \leq \begin{bmatrix} p^*p_1x_1 \\ p^*p_2x_2 \\ \cdot \\ \cdot \\ p^*p_n\,x_n \end{bmatrix}.$$

Thus $q_i \leq p^*$ for $1 \leq i \leq n$.

Similarly, if $p_* = \min_{1 \leq i \leq n} \{p_i\}$ then

$$A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n\,x_n \end{bmatrix} \geq p_* \begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n\,x_n \end{bmatrix} \text{ and } q_i \geq p_*,\ i = 1,2,\ldots,n.$$

Thus, (iii) follows from this lemma and the first part of the proof. This proves the theorem. ■.

Ideal Distributions

Let us go back now to positive matrices without concerning ourselves with improvement or any kind of order.

The Perron Frobenius Theorem

If A is any matrix of positive numbers, then:

1. The principal eigenvalue of A (the Perron Frobenius eigenvalue) is a positive real number whose corresponding eigenvector has positive entries. Therefore it can be regarded as a distribution. We express this distribution as [A]

2. Given any positive vector P, then lim[A^nP]---->[A] as n--->∞.Therefore [A^nP]~[A] for n large enough.

To see the significance of these results, let us suppose we are dealing with a person using the framework P,I,E,S and suppose that person is at the beginning of a momentous change in his life. He just got married, or he just began a profession, or he just got help from an addiction. Let us suppose that his uncertainties and fears have put him in state [P]. However, his desired state is [A], the percent of time he would really like to spend in the dimensions. Then activity A represents who he is , and by applying repeated applications of A to P, he will move closer and closer to his desired way of life. Of course, things will come up in his life where other momentous things will happen and he will be forced back to another vector Q, where he will apply powers of A to Q moving that vector closer to [A]. What is missing from making this a real life situation is that we need to control the pace of how the change will proceed. We can look at the history of the person and see when he should reach certain milestones in his progression.

Then we have to match it up to a matrix N which yields the proper progression, and where [N]=[A]. The following theorem allows us to do that.

Theorem 6. If A is any positive matrix whose main diagonal elements are a constant,"a"and whose principal eigenvalue is λ, then for k>0, kA has principal eigenvalue $k\lambda$ and a+j has principal eigenvalue λ+j,if it is >0.

Corollary 6.1 With either of these changes, all eigenvectors remain the same, and the eigenvector corresponding to the principal eigenvalue is the same eigenvector whose distribution is [A].

What this means is that we can make the off diagonal elements as close to 0 as we wish, and the diagonal elements as close to 1 as we wish. The closer the matrix comes to the identity matrix, the slower the change occurs.To do this in a formal way, we first set up a mold M. We call M a mold because it is a devise from which the matrices that we want can be obtained.

Let λ be the principal eigenvalue of A and let m be a number such that $m>\lambda$. Then 1/mA has equal diagonal elements d, 0<d<1. Let u=1-d. Multiplying the off diagonal elements of A by 1(mu), we

obtain m_{ij}=1/(mu)a_{ij},, $i \neq j$. Finally replace the diagonal elements by 1s. Thus the mold is

$$M=\begin{bmatrix} 1 & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21}. & 1. & m_{23} & \dots & m_{2n} \\ \\ m_{n1} & m_{n2}. & & \dots\dots. & 1 \end{bmatrix}$$

and the matrices are:

$$N(u)=\begin{bmatrix} 1-u & m_{12}.u. & m_{13}u \ldots\ldots\ldots & m_{1n}u \\ m_{21}u. & 1-u. & m_{23}u \ldots\ldots\ldots & m_{23}u \\ & & & . \\ m_{n1}.u. & m_{n2}u \ldots\ldots\ldots & & 1-u \end{bmatrix}$$

where $0<u<1$.

Notice that when $u=0$ we get the identity matrix where time literally stands still, nothing changes. It has no place where changeless objects do not exist. It represents the boundary of change. Likewise when $u=1$, the main diagonal elements are 0. It is still true that $[N(1)]=[A]$, since this is true for $u<1$, $u\neq0$. In fact , for any u, $-\infty<u<\infty$, $u\neq0$, N(u) will have the same eigenvectors. However, once we cross the boundary and go a short distance beyond, the principal eigenvector will not correspond to the positive eigenvector, so that when we take powers N^nP, negative numbers will come in some of the dimensions, since all the other eigenvectors of A have at least one negative entry.. Once that happens, the concept of a distribution vanishes. Since powers of N(u) applied to any vector P chase after the principal eigenvector, we need to confine ourselves to $0<u<1$.

Suppose we have an entity in a state P and a matrix N(u) which essentially matches the change in P as powers of N(u) are applied to P. Taking powers of N(u) and applying them to P is identical to the definition of the natural numbers from Alonzo Church's definition in the universal language of the λ calculus. In his version, N(u) is one dimensional. The trouble with this method in a changing universe is that, not only entities change constantly, but so do activities. To use only the activity N(u) over and over again is to miss the fact that N(u) changes continuously. To correct this oversight, we have the following theorem:

Theorem 7. If e=.0007, the equivalence limit in the P,I,E,S framework, then for u≤01 the matrix exp(nN(u)≅ e^nN^nP numerically.

Therefore [exp(nN)P]~[e^nN^nP]=[N^nP].

We see that when n is a natural number, [exp(nN)P]~[N^nP], u≤.01.We call such matrices equivalent pacewise. However, exp(xN) is defined for all x>0 and thus the matrix is changing continuously over time. Of course we lose a little of our cherished laws. 1+1≠ 2 since 1+1 corresponds to N(u)^2, and 2 corresponds to exp(2(N(u)). We can still say 1+1~2. But the very meaning of adding a 1 to a 1 assumes that after the first 1, the second 1 refers to the same unchanged object that no longer exists. Thus allowing a real numbered collection of matrices exp(xN(u)) is more realistic in a changing universe then simply using powers of a fixed activity N(u). In our example of a person in state P attempting to get to [A], we found, in one of our papers, that each application of N(u) takes 18 days. Now that we have a continuous collection of matrices, there is no reason to simply take powers of N(u). We can take exp(1/18N(u)) for our daily change, exp(1/432N(u)) for our hourly change, etc. We certainly have a much more realistic picture of how the person is changing over time as well as a method for finding fractional powers of any matrix.

If the diagonal entries are not all the same, we know that 1/mA gives us an activity matrix with the same ideal distribution [A]. Since it is an activity matrix, it has an improvement region $\mathcal{I}$ and a rate of improvement p. We also find that the main diagonal elements d_i are all between 0 and 1. If we let $u_i=1-d_i$ and let $u= (\prod u_i)$^1/n, the geometric mean, then we can consider u to be the pace of the matrix.

From this particular matrix however, we can find a mold M that gives rise to a set of matrices all having the same $\mathcal{I}$, p, and u.

We first divide each off diagonal element in row i by $1/u_i$ and replace the diagonal elements by 1s. Thus from $A(a_{ij})$) we get $m_{ij}=(1/m \cdot u_i)a_{ij}$ $i \neq j$.

However, if we consider all possible numbers u_i $0<u_i<1$, with geometric mean u, we find that the set of all such matrices corresponds to every sort of person in the population including 1/mA.

N(u) represents the normal individual in our society where the size of u determines the pace of change. Whatever u is, we can obtain, from the mold M, the set of all matrices

$$B(u_1,u_2,\ldots.u_n)=\begin{bmatrix} 1-u_1 & m_{12}u_1 & m_{13}u_1\ldots & .m_{1n} \\ m_{21}u_2 & 1-u_2. & m_{23}u_2\ \ldots, & m_{2n}u_2 \\ & & & \\ m_{n1}u_n. & m_{n2}u_n.\ \ldots & \ldots\ldots & 1-u_n \end{bmatrix}$$

where u is the geometric mean of the u_i's. We call this set Pop(u), the population with respect to u.

If we want to change u to u', multiplying each u_i by u'/u changes B to B' where

[B]=[B'], $u_i<1$, I=1,2,3,......,n

Since Pop(u) consists only of activity matrices we can think of u, the pace of the matrix as either representing the pace of change if we are considering the matrix as moving the person toward his ideal way of life, or we can consider the pace u to represent the pace of improvement as he moves toward his goals.

When we previously considered N(u), we were concerned with the person moving toward [A]. We used theorem7. for fractional powers of the matrix. But this took us away from activity matrices. We first want to look at Pop(u) from this point of view.

Theorem 7 applies to all matrices in Pop(u_0), $u_0 \leq .01$. We also have the following Theorem connecting A(u) with A(u_0):

Theorem 8. For $u>u_0$, e^((ku/u_0)-k)exp(kA(u)P≅exp(ku/u_0A(u_0)P

Therefore, for any matrix A in Pop(u),

[exp(t(A(u)P)~[A(u_0)^kP], where k=tu/u_0 for any positive integer k.

What this shows is that we can use powers of matrices in Pop(u_0) that are equivalent pacewise with particular powers of t.

Looking at matrices the other way, the off diagonal elements of the M matrix for Pop(u) identifies the bounding vectors V_i of the improvement region. This comes from the fact that the cofactors of I-A are the entries of the V_is. We can then use induction on n to show:

Every matrix in Pop(u) has the same improvement region.

Furthermore since |I-A|= $\prod u_i C(m_{ij})$, we see that if $\prod u_i$=u^n, a constant, then the rate of improvement is also fixed. Thus:

Every matrix in Pop(u) has the same rate of improvement.

Because of these properties of Pop(u), we can state:

Axiom 1. Every person in our population can be associated with a matrix in Pop(u) which represents his instinctive drives.

What this means is that for each of us there is an ideal distribution of our time among the dimensions, and given no obstacle, it is where we want to be.

It represents who we are. An activity matrix A where [A] is the ideal distribution for a person, is called his identity matrix or ID matrix.

. We see that an ID matrix, operating on any vector can, at any stage, be looked at in two different ways simultaneously. If we look at the numerical values of the vector, they eventually decrease toward the origin as powers of the matrix keep increasing.At the same time, the state of the vectors keep moving toward [A], the ideal of how we want to live. Since the origin represents the common goals in our society, we can see how close we are to our goals and compare ourselves to others,driven by our egos, or we can concentrate on moving toward our ideal way of living, driven by our better angels, toward a richer, more fulfilling way of life for ourselves. What makes [A] so different from the origin, is that it is ideal only for one person. Any other person will have a different ID matrix and a different ideal. There is no competition, no better or worse, no richer or poorer, etc..

Local Improvement

To recapitulate, we started with an arbitrary positive matrix A. We then found an activity matrix with the same ideal distribution as A, having an improvement region $\mathcal{I}$, a rate of improvement p, and a pace u. We then found all activity matrices with the same $\mathcal{I}$, p, and u,

However, if a person whose ideal distribution is [A], is in a state P which is not in $\mathcal{I}$, it could take many applications (many years) before he begins to improve. We see examples of that when a person moves to a foreign country, unable to speak the language and spends an extraordinary length of time adjusting to the culture before being able to improve like those around him.

For the most part, however, we remain in our own country and try to engage in activities that move us closer to our common goals. We will now deal with a neighborhood of the person in state P that we want included in $\mathcal{I}$,

Given two points Pw and Pb representing the worst and best scenarios that we wish to consider, where Pb<P<Pw. This set gives rise to a unique cone stemming from the origin containing that set. This can easily be illustrated in two dimensions(see figure 1.). The set of points betweenPb and Pw is the shaded region. All cones representing improvement regions are bounded by two rays starting at the origin.

Obviously there is a smallest cone containing the set whose bounding rays pass through (b_1,w_2) and (w_1,b_2). We want all matrices that have this improvement region.

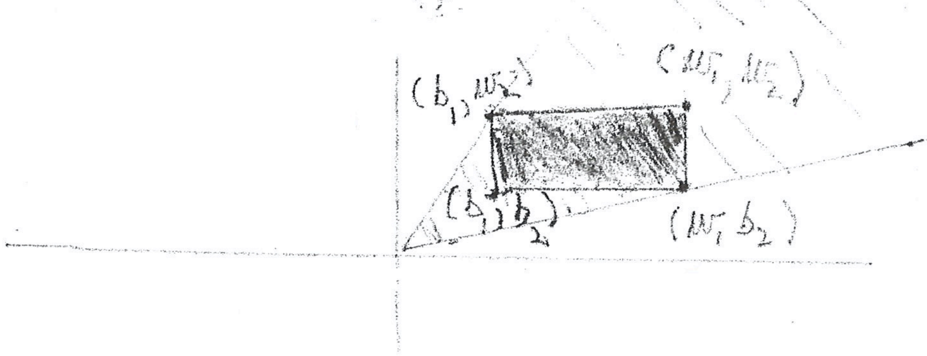

Figure 1.

Turning now to the n-dimensional case we shall construct a formula that yields all such matrices.

At first, suppose we are given points

$$P_W = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \text{ and } P_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \qquad 0 < \alpha_i < 1.$$

We first construct a matrix A^* that has an improvement region induced by P_W and P_B. Let

$$A^* = \begin{bmatrix} k_1 & k_1 & \cdots & k_1 \\ k_2 & k_2 & \cdots & k_2 \\ \cdot & & & \\ \cdot & & & \\ k_n & k_n & \cdots & k_n \end{bmatrix} \qquad 0 < k_i < \frac{1}{n} \quad i = 1, 2, \ldots, n.$$

Then

$$A^* \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

and A^* has an improvement regions $I_{A^*} \neq \phi$. We shall show that for special values of the k_i, I_{A^*} is induced by P_W and P_B.

Vertices of the region between P_W and P_B are of the form

$$P_j = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \alpha_i \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad j = 1,2,\ldots,n.$$

Furthermore, since every point in Γ_{A^*} is of the form

$$\sum_{i=1}^{n} c_i V_i^*, \qquad c_i \geq 0, \qquad V_i^* = \begin{bmatrix} U_{i1}^* \\ \cdot \\ \cdot \\ U_{in}^* \end{bmatrix}$$

where U_{ij}^* are the cofactors of $I - A^*$; and, since each P_j lies on the hyperplane with basis $V_1^*, V_2^*, \ldots, V_{j-1}^*, V_{j+1}^*, \ldots, V_n^*$, we have

$$P_j = \sum_{i=1}^{n} c_{ji} V_i^* \qquad \text{where } c_{jj} = 0. \tag{6}$$

We calculate V_i^* from the matrix $I - A^*$. Thus

$$I - A^* = \begin{bmatrix} 1-k_1 & -k_1 & \ldots & -k_1 \\ -k_2 & 1-k_2 & \ldots & -k_2 \\ \cdot & & & \\ \cdot & & & \\ -k_n & -k_n & \ldots & 1-k_n \end{bmatrix}$$

From this matrix it is readily verified that

$$V_i^* = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ k_{i-1} \\ 1 - \sum_{j \neq i} k_j \\ k_{i+1} \\ k_n \end{bmatrix}$$

Therefore (6) leads to the system of $n-1$ linear equations in unknowns $c_{ji},\ i = 1,2,\ldots,n,\ i \neq j$.

Solving this system, we obtain by Cramer's rule

$$c_{ji} = \frac{1 - k_j - (n-1)k_i}{\left[1 - \sum_{l=1}^{n} k_l\right](1 - k_j)}. \tag{7}$$

Note that because of the restriction on the k's, the determinant of the coefficients is greater than 0. Using the j'th equation $\sum_{i \neq j} c_{ji} k_j = \alpha_j$ and substituting for the c_{ji}'s in (7) we obtain

$$\sum_{i \neq j} c_{ji} k_j = \frac{\sum_{i \neq j} [(1-k_j)-(n-1)k_i]}{\left[1-\sum_{i=1}^{M} k_i\right](1-k_j)} = \frac{(n-1)k_j}{1-k_j} = \alpha_j .$$

Thus, solving for k_j we get

$$k_j = \frac{\alpha_j}{n-1+\alpha_j} \qquad j = 1,2,\ldots,n. \tag{8}$$

NOTE: The i'th coordinate α_i of P_i remains invariant under A^* *i.e.*, $A^* P_i$ has i'th coordinate α_i.

To see this, it is only necessary to observe that the i'th coordinate of $A^* P_i$ is

$$(n-1)k_i + k_i \alpha_i = k_i(n-1+\alpha_i) = \frac{\alpha_i}{n-1+\alpha_i}(n-1+\alpha_i) = \alpha_i .$$

Moving on to the general case, suppose

$$P_W = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \text{ and } P_B = \begin{bmatrix} \alpha_1 w_1 \\ \alpha_2 w_2 \\ \cdot \\ \cdot \\ \alpha_n w_n \end{bmatrix} \qquad 0 < \alpha_i < 1, \ i = 1,2,\ldots,n.$$

We can reduce this to the previous case by using the matrix

$$A^* = \begin{bmatrix} \frac{w_1}{w_1}k_1 & \frac{w_1}{w_2}k_1 & \cdots & \frac{w_1}{w_n}k_1 \\ \frac{w_2}{w_1}k_1 & \frac{w_2}{w_2}k_2 & \cdots & \frac{w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{w_n}{w_1}k_1 & \frac{w_n}{w_2}k_2 & \cdots & \frac{w_n}{w_n}k_n \end{bmatrix}.$$

Going through an analogous argument, we arrive at the same expression (8) for the k_j's.

Now

$$I-A^{*}=\begin{bmatrix} 1-\frac{w_1}{w_1}k_1 & \frac{-w_1}{w_2}k_1 & \cdots & \frac{-w_1}{w_n}k_1 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}k_1 & & \cdots & 1-\frac{w_n}{w_n}k_n \end{bmatrix}.$$

If we multiply each element in the i'th row of U^{*} by a positive number l_i, we get a scalar multiple of V_i in the same direction. Thus, this set of vectors generates the same improvement region.

Let

$$U=\begin{bmatrix} l_1(1-k_1) & l_1\frac{w_1}{w_2}k_1 & \cdots & -l_1\frac{w_1}{w_n}k_1 \\ -l_2\frac{w_2}{w_1}k_2 & l_2(1-k_2) & \cdots & -l_2\frac{w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ -l_n\frac{w_n}{w_1}k_1 & & \cdots & l_n(1-k_n) \end{bmatrix}$$

If we now define A so that $U=I-A$ then $A=I-(I-A)$ and therefore

$$A=\begin{bmatrix} 1-l_1(1-k_1) & \frac{w_1}{w_2}l_1k_1 & \cdots & \frac{w_1}{w_n}l_1k_1 \\ \frac{w_2}{w_1}l_2k_2 & 1-l_2(1-k_2) & \cdots & \frac{w_2}{w_n}l_2k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{w_n}{w_1}l_nk_n & \frac{w_n}{w_2}l_nk_n & \cdots & 1-l_n(1-k_n) \end{bmatrix} \tag{9}$$

The l's are not any positive numbers, but since the main diagonal elements of A must be numbers between 0 and 1, we must have

$$0\le 1-l_i(1-k_i)<1$$

or

$$l_i\le\frac{1}{1-k_i}\qquad i=1,2,\ldots,n. \tag{10}$$

Now consider the matrix

$$B = \begin{bmatrix} (1-k_1) & \frac{-w_1}{w_2}k_1 & \dots & \frac{-w_1}{w_n}k_1 \\ \frac{-w_2}{w_1}k_2 & (1-k_2) & \dots & \frac{-w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}k_n & \frac{-w_n}{w_2}k_n & \dots & (1-k_n) \end{bmatrix}$$

It can be readily shown that

$$\det B = 1 - \sum_{i=1}^{n} k_i$$

But

$$I - A = \begin{bmatrix} l_1(1-k_1) & \frac{-w_1}{w_2}l_1k_1 & \dots & \frac{-w_1}{w_n}l_1k_1 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}l_nk_n & & \dots & l_n(1-k_n) \end{bmatrix}$$

and therefore

$$\chi(1) = \det(I - A) = \prod_{i=1}^{n} l_i (\det B) = \prod_{i=1}^{n} l_i [1 - \sum_{i=1}^{n} k_i]$$

Thus

$$p = 1 - \prod_{i=1}^{n} l_i \left[1 - \sum_{i=1}^{n} k_i\right]. \tag{11}$$

Since $l_i \le \frac{1}{1-k_i}$, we have

$$\prod_{i=1}^{n} l_i \le \prod_{i=1}^{n} \frac{1}{1-k_i}.$$

Hence

$$p \ge 1 - \prod_{i=1}^{n} \left[\frac{1}{1-k_i}\right] + \frac{\sum_{i=1}^{n} k_i}{\prod_{i=1}^{n}(1-k_i)}$$

or

$$\frac{\prod_{i=1}^{n}(1-k_i) + \sum_{i=1}^{n} k_i - 1}{\prod_{i=1}^{n}(1-k_i)} \le p < 1. \tag{12}$$

In order to help simplify the following results, we shall introduce some new variables.

Let $\eta_i = l_i(1-k_i)$, $i = 1,2,\ldots,n$ and $\eta = \left[\prod_{i=1}^{n} \eta_i\right]^{1/n}$. Note that η is the geometric mean of $\eta_1, \eta_2, \ldots, \eta_n$. We shall also let

$$s_0 = 1, \quad s_j = \frac{\sum_{i_1<i_2<\ldots<i_j} (1-k_{i_1}-k_{i_2}-\ldots-k_{i_j})\eta_{i_1}\eta_{i_2}\ldots\eta_{i_j}}{\prod_{l=1}^{j}(1-k_{i_l})}, \quad j = 1,2,3,\ldots,n. \tag{12}$$

Finally, let

$$\eta_i = \varepsilon_i \eta. \tag{13}$$

It is well known that

$$X(x) = \sum_{k=0}^{n} (-1)^k S_k x^{n-k} \tag{14}$$

where $S_k =$ sum of the principal minors of A of order k (see [6]), p. 198, for example).

Letting A be given by (9), we see that

$$S_j = \sum_{i=0}^{j-1}(-1)^{i+n}(n-i)s_i + (-1)^{i+n}s_j. \tag{15}$$

Replacing (15) into (14) and rearranging terms, we obtain

$$X(x) = \sum_{i=0}^{n} s_i (x-1)^{n-i}. \tag{16}$$

Theorem 9. *If all the k's have the same value, then the matrix with the smallest λ will be that unique matrix all of whose main diagonal entries are equal. In this case*

$$\lambda = 1 - \frac{1-nk}{1-k}\eta.$$

Proof. From (16) we write $X(x) = \sum_{i=0}^{n} s_i(x-1)^{n-i}$ where, under the assumption of equal k's, we can write

$$s_i = \frac{1-ik}{(1-k)^i}\left[\sum_{j_1<j_2<\ldots<j_i} \varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_i}\right]\eta^i.$$

We keep x fixed between 0 and 1, and k fixed, and consider X as a function of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$, since

$$\varepsilon_n = \frac{\eta^n}{\prod_{i=1}^{n-1}\varepsilon_i}.$$

We seek to find the critical points of X. Taking first order partials, we obtain after simplification

$$\frac{\partial X}{\partial \varepsilon_i} = (x-1)\left[\prod_{\substack{j=1 \\ j\neq 1}}^{n-1} \varepsilon_j \varepsilon_i^2 - 1\right]\eta P_i(x), \tag{*}$$

where

$$P_i(x) = (x-1)^{n-2} + \frac{1-2k}{(1-k)^2}\left[\sum_{\substack{j=1 \\ j\neq 1}}^{n-1}\varepsilon_i\right](x-1)^{n-3}\eta + \frac{1-3k}{(1-k)^3}\left[\sum_{\substack{j_1,j_2=1 \\ j_1,j_2\neq i \\ j_1\neq j_2}}^{n-1}\varepsilon_{j_1}\varepsilon_{j_2}\right](x-1)^{n-4}\eta^2$$

$$+\ldots+\frac{1-(n-1)k}{(1-k)^{n-1}}\prod_{\substack{j=1 \\ j\neq i}}^{n-1}\varepsilon_j\eta^{n-2}.$$

Setting $\frac{\delta X}{\delta\varepsilon_i}=0,\ i=1,2,\ldots,n-1$, we have either $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=1$ or, in order that all $P_i(x)=0$, we must have $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=\varepsilon$.

Lemma 9.1. *Let $X_\varepsilon(x)$ be the characteristic polynomial where $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=\varepsilon$. Then, if $x_\varepsilon=1-\frac{\varepsilon}{1-k}\eta$,*

$$X_\varepsilon(x)=(x-x_\varepsilon)^{n-2}\left\{(x-1)^2+\left[1-\frac{(n-1)k}{1-k}\varepsilon+\frac{1}{\varepsilon^{n-1}}\right]\eta(x-1)+\frac{1}{\varepsilon^{n-2}}\frac{1-nk}{(1-k)^2}\eta^2\right\}.$$

In particular, this implies that x_ε is a root of multiplicity $n-2$.

This can be proved by induction.

Corollary 9.1.

$$\lambda_\varepsilon=1-\left\{\frac{1}{2}\left[\frac{1-(n-1)k}{1-k}\varepsilon+\frac{1}{\varepsilon^{n-1}}\right]\eta-\frac{1}{2}\sqrt{\frac{1-(n-1)k}{(1-k)}\varepsilon+\left[\frac{1}{\varepsilon^{n-1}}\right]^2-\frac{4(1-nk)}{\varepsilon^{n-2}(1-k)^2}\eta}\right\}$$

This comes from using the quadratic formula on the expression in braces and selecting the largest root.

Corollary 9.2. *x_ε is a root of multiplicity $n-3$ of $\frac{dX_\varepsilon}{d\varepsilon}$.*

Lemma 9.2. *$\bar{x}_\varepsilon=1-\frac{1-(n-1)k}{(1-k)^2}\varepsilon\eta$ is a root of $\frac{dX_\varepsilon}{d\varepsilon}$.*

This can be shown by direct substitution. Since $P_i(x)$ is of order $n-2$, the only critical values of X are x_ε and $\bar{x}_\varepsilon$. Furthermore, $x_\varepsilon<\bar{x}_\varepsilon\le\lambda_\varepsilon$ as can readily be shown.

Lemma 9.3.

$$\left[\frac{1-P}{1-nk}\right]^{\frac{n-1}{n}}(1-k)^{n-1}\le\varepsilon_i\le\left[\frac{1-nk}{1-P}\right]^{1/n}\frac{1}{1-k}.$$

The proof comes about in a straightforward fashion from $l_i\le\frac{1}{1-k}$, $i=1,2,\ldots,n$ and $P=1-\prod_{i=1}^{n}l_i[1-nk]$.

Now $X_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-2},\varepsilon_{n-i}} - X_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-1}}$ for some $i \neq n-1$

$$= \frac{(x-1)\left(\prod_{j=1}^{n-2}\varepsilon_j\varepsilon_{n-1}^2 - 1)(\varepsilon_{n-1}-\varepsilon_{n-1}\right)}{\prod_{j=1}^{n-2}\varepsilon_i\varepsilon_{n-1}^2} P_{n-1}(x).$$

Lemma ?.4. *Let* $\bar{x}_{\bar{\varepsilon}} = 1 - \frac{1-(n-1)k}{(1-k)^2}\bar{\varepsilon}\eta$. *Then*

$$P_{n-1}(\bar{x}_{\bar{\varepsilon}}) = \sum_{l=1}^{n-3} a_l \sum_{C_l} \prod_{i=1}^{l}(\varepsilon_{j_i} - \bar{\varepsilon}) + a_{n-2}\left[\prod_{j=1}^{n-2}\varepsilon_j - \bar{\varepsilon}^{n-2}\right],$$

where $\sum_{C_l}$ *is over all combinations of* ε_j *taken* l *at a time and where the* $a_i = a_i(k,\bar{\varepsilon}) > 0$.

This can be proved by induction on n.

Lemma ?.5. $X(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-1})$ *has a maximum at* $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n-1} = 1$ *for all* x *such that* $x \geq 1 - \frac{1-(n-1)k}{(1-k)^2}\eta$. *Furthermore it is the only maximum in that range of* x.

Proof. First note that

$$\frac{\partial X}{\partial \varepsilon_i} = (x-1)\frac{\prod_{\substack{j=1\\ j\neq 1}}^{n-1}(\varepsilon_j\varepsilon_i^2 - 1)\eta}{\prod_{j=1}^{n-1}\varepsilon_j\varepsilon_i^2} P_i(x),$$

where $P_i(x)$ is defined by (*). For $\varepsilon_i = 1$, $i = 1,2,\ldots,n-1$, we see that $\frac{\partial X}{\partial \varepsilon_i} = 0$ regardless of $P_i(x)$. If all the ε_i's are not 1, then $\frac{\partial X}{\partial \varepsilon_i} = 0$ only when all $P_i(x) = 0$ and that occurs only when $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n-1} = \varepsilon$ and $x = 1 - \frac{1-(n-1)k}{(1-k)^2}\varepsilon\eta$. If $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n-1} = \varepsilon$, then we denote $X(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-1})$ by X_ε. If $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n-2} = \varepsilon$, $\varepsilon_{n-1} = \sigma$, we denote $X(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-1})$ by $X_{\varepsilon,\sigma}$.

But, from Lemma .4, if $\varepsilon < 1$, then

$$X_\varepsilon(\bar{x}_\varepsilon) > X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma < \varepsilon$$

$$X_\varepsilon(\bar{x}_\varepsilon) < X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma > \varepsilon.$$

This means that $(\varepsilon,\varepsilon,\ldots,\varepsilon,\bar{x}_\varepsilon)$ is a saddle point for $\varepsilon < 1$.

Likewise, for $\varepsilon > 1$,

$$X_\varepsilon(\bar{x}_\varepsilon) < X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma < \varepsilon$$

$$X_\varepsilon(\bar{x}_\varepsilon) > X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma > \varepsilon.$$

Thus $(\varepsilon, \varepsilon, \ldots, \varepsilon, \bar{x}_\varepsilon)$ is a saddle point for all $\varepsilon \neq 1$. However, for $\varepsilon = 1$, $(1, 1, \ldots, 1, \bar{x}_1)$ is a maximum. Furthermore $X_1(\bar{x}_1) < 0$.

Since the ε_i's are bounded in a closed interval from Lemma 6.3, and since $f(X_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}) = \lambda_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}$ is a continuous function, it attains its maximum and minimum in that interval. Since maximum λ is obtained on the boundaries of the ε_i as can readily be shown, we can see that $x_{1,1,\ldots,1}$ is the only polynomial that maximizes all of the others in some neighborhood around it and has a smaller λ than any of the others.

This completes the proof of Theorem 9

When the k's are not all the same, the question arises whether the matrix, all of whose main diagonal elements are the same, is still the "best matrix", *i.e.* the one with smallest λ. The answer to this is "no" as will be shown.

Since all characteristic polynomials $X(x)$ in the equivalence class $\varepsilon(I_{A,P})$ pass though the point $(1, X(1))$ and have positive slopes at that point, one can ask for the polynomial whose slope is a minimum, since that curve will be above all the others, at least in a neighborhood of that point. In order to find this matrix, we first find the slope of $X(x)$ for $x = 1$, *i.e.*,

$$\left.\frac{\partial X}{\partial x}\right]_{x=1} = S_{n-1}.$$

Theorem 10 *The matrix whose slope at* $(1, X(1))$ *is a minimum is the matrix where*

$$\eta_i = \frac{\eta}{\delta}\,\delta_i \quad i = 1, 2, \ldots, n,$$

where

$$\delta_i = \left[1 - \sum_{j \neq i} k_j\right](1 - k_i) \text{ and } \delta = \left[\prod_{i=1}^{n} \delta_i\right]^{1/n}.$$

Proof. It can be readily be shown that

$$s_{n-1} = \frac{\eta^n}{\prod_{j=1}^{n}(1-k_j)} \left[\sum_{j=1}^{n-1} \frac{\delta_j}{\eta_j} + \frac{\delta_n}{\eta^n} \prod_{j=1}^{n-1} \eta_j \right].$$

Then

$$\frac{\partial s_{n-1}}{\partial \eta_i} = \frac{\eta^n}{\prod_{j=1}^{n}(1-k_j)} \left[\frac{-\delta_i}{\eta_i^2} + \frac{\delta_n}{\eta^n} \prod_{\substack{j=1 \\ j \neq 1}}^{n-1} \eta_j \right] = 0$$

$$\frac{\delta_i}{\eta_i^2} = \frac{\delta_n}{\eta^n} \prod_{\substack{j=1 \\ j \neq 1}}^{n-1} \eta_j \frac{\eta_i \eta_n}{\eta_i \eta_n} = \frac{\delta_n}{\eta^n} \frac{\eta^n}{\eta_i \eta_n} = \frac{\delta_n}{\eta_i \eta_n}$$

or $\frac{\delta_i}{\eta_i} = \frac{\delta_n}{\eta_n}$ for $i = 1,2,\ldots,n-1$. This equation obviously holds for $i = n$ as well. Therefore $\prod_{i=1}^{n} \frac{\delta_i}{\eta_i} = \left[\frac{\delta_n}{\eta_n}\right]^n$ which implies $\frac{\delta}{\eta} = \frac{\delta_n}{\eta^n}$. Thus, $\frac{\delta_i}{\eta_i} = \frac{\delta}{\eta}$ or $\eta_i = \frac{\eta}{\delta}\delta_i$.

This completes the proof.

Denoting the matrix whose main diagonal elements are equal by A_e, the one whose characteristic polynomial has a minimum slope at $(1, X(1))$ by A_m and the matrix with smallest λ by A^*, it can readily be shown that in two dimensions

$$A_e = A_m = A^*.$$

However, in three dimensions, this is not the case.

Let

$$X_e = (x-1)^3 + s_1(x-1)^2 + s_2(x-1) + s_3$$

and

$$X_m = (x-1)^3 + s'_1(x-1)^2 + s'_2(x-1) + s'_3.$$

Since $s_3 = s'_3$, $s'_2 < s_2$ by definition of X_m and $s < s'_1$, the geometric means being less than the arithmetic means, we have

$$X_e - X_m = (x-1)[(s_1 - s'_1)(x-1) + (s_2 - s'_2)] < 0 \quad 0 < x < 1.$$

Thus $X_e < X_m$ for $0 < x < 1$ and so when $X_m(\lambda_m) = 0$, $X_e(\lambda_m) < 0$ implying that $\lambda_m < \lambda_e$. We see that in three dimensions, A_m is better than A_e. However, it is an open question whether it is the "best". For dimensions greater than three, we do not even know that much.

Going back to (9), letting $u_i = \eta_i = l_i(1-k_i)$, we have $l_i = u_i/(1-k_i)$ and $l_ik_i = (k_i/(1-k_i)u_i$. Letting $s_i = k_i/(1-k_i)$, we can set up a mold as follows:

$$M = \begin{bmatrix} 1 \quad w_1/w_2s_1 \quad w_1/w_3s_1 \ldots\ldots\ldots\ldots w_1/w_ns_1 \\ w_2/w_1s_2 \quad 1. \quad w_2/w_3s_2 \ldots\ldots w_2/w_ns_2 \\ . \\ w_n/w_1s_n.. \quad w_n/w_2\,s_n. \quad w_n/w_3s_n \ldots\ldots.. 1 \end{bmatrix}$$

We call such a mold a local model, since it has special properties not shared by a general mold.

Theorem 11. If M is a local model, then all the eigenvalues for any matrix in Pop(u) are real numbers.

Proof. We can use induction to show that the characteristic polynomials are independent of the w's. Hence, letting $(w_i=u_1s_1/u_is_i)$^1/nw_1, l=1,2,.....n, we obtain a symmetric matrix with the same characteristic polynomial.

Since the eigenvalues of symmetric matrices are real, so are the eigenvalues of our given matrix.

This result is not true for molds in general. We certainly can have complex roots. Therefore it is necessary to distinguish between molds in general and local models.

From M, we obtain our matrices

$$A(u_1,u_2\ldots u_n) = \begin{bmatrix} 1-u_1 \quad w_1/w_2s_1u_1. \quad w_1/w_3s_1u_1 \ \ldots . \ w_1/w_ns_1u_1 \\ w_2/w_1s_2u_2 \quad 1-u_2 \quad w_2/w_3s_2u_2. \ldots.. \ w_2/w_ns_2u_2 \\ \ldots \\ \ldots \\ w_n/w_1s_nu_n. \quad w_n/w_2s_nu_n \quad w_n/w_3s_nu_n. \ldots \ldots \ 1-u_n \end{bmatrix}$$

When $\prod u_i = u^n$, we then have a collection of matrices with a common $\mathcal{I}$, p, and u,whtch we call Pop(u). Only this time every matrix in Pop(u), improves the person from the beginning.

When we think of a normal person in our society, we think of someone engaging right away with activities that move him toward the goals of society. We also do not expect the normal person to spend any excessive time in any one particular dimension. Thus we expect the pace in each coordinate to be the same. But this is just N(u), which we take as the ideal distribution of the normal individual. We also assume that one of the matrices is the ID matrix of the person.

Since we can decide on Pw and Pb, it gives us control of what the improvement region will look like . The local model is the preferred mode and has been used in most of our papers.

Matrix Applications

1. Marriage

Whatever happens to us in life, our ID matrix A is always with us. If one gets married to a person whose ID matrix is B, then their way of life together will be BA ,and will have their ideal as [BA]. If A and B have ideals close together then [BA] will be close to [A] and to [B]. These people can claim truthfully that they have found their soul mates. The marriage should go well. If [A] and [B] are far apart, then [BA] will not be close to [A] or [B] and there will be frustration on both their parts that they are not going where they ultimately want to be. However they may have the same goals, and striving together toward them, may be sufficient to keep the marriage together. Love is also a powerful force and if continued strongly, may be worth not getting close to one's ideal.

2. Business

Not only does Pop(u) represent all types of people in our population, it also represents the variety of ways a company of n stores as its framework can redistribute its time and money to maximize its profits. This involves choosing the best matrix from Pop(u). In this framework, we first start with a vector $G(g_1,g_2\ldots g_n)$ which represents the goal. The goal values give us the size of the business.These values represent the maximum possible amount that any store can expect to make given that there is no expansion or downsizing that would change the nature of that store. Rather there are so many hours in a day, and so many employees who work there, etc, and given a certain sized business, there certainly is an upper limit to what can be expected. In standard form G is the origin. Since all our matrices have the same improvement region, rate of improvement, and pace, they are variations of the same business. Where they differ is in the principal eigen value and its corresponding eigenvector.The matrix with the smallest principal eigenvalue will eventually get closer to the goal than any other matrix in Pop(u), and once that happens it will maintain and increase its lead as time goes on. One of the major problems here is finding this matrix.

3. Disease

We return now to the P,I,E,S, framework in discussing disease. The ID matrix is always with the person, so that the dysfunction must come from a matrix combining with the ID matrix that reflects the problem. The matrices we use to do this are the inverse matrices of Pop(u). We can always change a matrix by perturbation that renders it invertible. Given any two distinct matrices in Pop(u), the matrix B^-1A will be such that one of the coordinates will so dominate that almost all time will go into that coordinate. Life will be unsustainable. In general, the P coordinate will dominate with diseases such as cancer, heart problems, emphysema, etc. The I coordinate dominates in ALS, where the other dimensions use just a small amount of time to sustain life, We have such an example in the life of Steven Hawking.whose intellect soared throughout his life.. Emotions dominate with phobias,,PTSD, panic attacks, etc, while Spirituality increases beyond reason in insanity, schizophrenia, paranoia, etc.

We run into trouble when we apply the inverse matrix with itself, A^-1A=I, getting to the identity matrix where time stands still. Also, for matrices A', close to A, A'^-1A takes a great amount of time before there is any real movement. For these reasons, there is a much better way to model diseases as well as obtain whatever pace we need to mimic the actual progression.

Theorem 12 Let $A(u_1,u_2,,u_3,u_4)$ be an activity matrix and let q be a positive number . such that qu_i and $1/qu_i$ are between 0 and1, For any two coordinates, we replace u_i with l/q_iu_i and u_j with qu_j . Calling the new matrix A', we have A'^-1A—-> ith coordinate and A^-1A'——> jth coordinate.

Corollary 12.1The closer q is to 1, the slower the pace.

We see again how the pace depends on how close we come to I. Not only can we control the pace of the disease, but we can also control which coordinate will dominate. This means that we can model any disease so far as the coordinates are concerned.by by replacing any coordinate u_i by$1/qu_i$. We can work with chronic diseases that are not fatal, and if fatal, how long a person has to live , using his past progression . Even aging can be regarded as a fatal disease since we all have to
die sometimes.

Notice also , if M is a local model, then all the inverse matrices also have real eigenvalues.

4. Science

The identity matrix I, which occurs when u=0 is certainly a singularity in the sense that powers of I applied to any vector do not move toward an eigenvector but rather keep every vector fixed, displaying the fact that time literally stands still.

If we think of the evolution of the universe, we do not think of it as improving toward 'a goal. Rather, it is moving toward an ideal, which could be forever separating, eventually coming together, or eventually fixing on some steady state. In any case we can imagine I as empty space.. It can also be thought of as containing A^-1A for all A in Pop(u). The fact that A`^-1A and A^-1A' tend to two different dimensions regardless how close to 1 we take q, shows us that with ever so slightly a difference, bifurcation occurs, which is exactly what symmetry breaking entails. We can think of empty space as seething with A^-1A matrices,cancelling one another; but every so often, breaking the symmetry and having an element A or an anti-element A^-1move toward its eigenfunction. Furthermore, the slightest breeze so to speak, perhaps an observation, will push A'^-1A toward one of its two choices. It mimics quantum mechanics.

As u increases,, all matrices in Pop(u) will move vectors to their positive eigenvectors changing more rapidly as u increases in value until we get close to another singularity. This occurs when λ and $-\lambda$ are the largest and smallest eigenvalues i.e.$|\lambda|$ is a double root. When that value if u is realized, powers of A do not tend to any eigenvector of A. Therefore we have another singularity. We can readily find that value of u by the following algorithm:

Starting from some u_0, $A(u_0)$ and letting λ_1 be the largest eigenvector and λ_s be the smallest, we find $\lambda_l(.1)$, $\lambda_s(.1)$, $\lambda_l(.2)$, $\lambda_s(.2)$, and $\Delta\lambda_I = \lambda_I(,1) - \lambda_I(.2)$, i=ls.
Then

$$u_s = (.1)[(\lambda_l(.1) + \lambda_s(.2))/(\Delta(.1) + \Delta_s(.2))] + .1$$

As u approaches u_s, while still remaining less than u_s, powers of A(u) will still tend to the positive eigenvector, but the closer you get to u_s the longer it takes to get toward the ideal; in fact, it keeps oscillating between two distributions and only after an enormous amount of time do they close in on their positive eigenvector, mimicking approaching the event horizon. where the closer you get to a black hole, time approaches infinity.

Instead of u representing how fast we are moving toward the eigenvector, it is now slowing down to time almost stopping as is the case with I.

When we actually get inside the black hole I.e. $u=u_s$, all the powers of A applied to any vector keep oscillating between two distributions depending on whether the power is even or odd.

When $u>u_s$, the negative eigenvalue dominates and each vector tends toward the corresponding eigenvector except the the sign keeps changing as we move from odd to even powers.

When $u<0$, the principal eigenvector is positive, so there is no sign change as we approach it's eigenvector.

One final thing to note is that throughout this paper; either u represents how quickly one either improves or changes. If we regard u as change, we see that if we change so quickly that we approach the event horizon, we will keep oscillating between two distributions and only minutely move toward the positive eigenvector. The person's behavior models that of someone with bipolar disorder. It would seem to indicate, by mathematics alone that there is a limit to how quickly one can change before it becomes a serious disorder.

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