The Best Matrix

by

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In [1], p.24, while searching for the matrix A* formed from a local model with the smallest principal eigenvalue λ, we considered two possibilities: The matrix Ae with equal main diagonal elements, and Am, the matrix whose characteristic polynomial $\varkappa(x)$ has minimum slope at $(1, \varkappa(1))$. It can easily be shown that in two dimensions Ae=Am=A*.

However, in three dimensions this is not the case. In fact we showed in [1] that $Am < Ae$ in the sense that Am has a smaller eigenvalue than Ae. However, it was an open question whether it is the best. For dimensions greater than three, we did not even know that much. That is, until we finally obtained the minimax algorithm which is the content of this paper. Using it, we can definitely say that neither Ae or Am is the best matrix except in very special cases.

The Minimax Algorithm

Theorem 6 in [1] specifies that if all the k"s are equal, then the best matrix is Ae. This will be our starting point. We will next consider the local model M defined in [2] on p.38:

$$M=\begin{bmatrix} 1 & w1/w2s1 & w1/w3s1 & \ldots & w1/wns1 \\ w2/w1s2 & 1 & w2/w3s2 & \ldots & w2/wns2 \\ . & & & & \\ . & & & & \\ wn/w1sn & wn/w2sn & \ldots & & 1 \end{bmatrix}$$

where si=ki/(1-ki) and mij=wi/wjsi, I=12...n. Notice that mijmji=sisj which depends only on the k"s. If all the k"s are equal, then Ae(u) is the best matrix for any u, 0<u<1.

If the k"s are not all the same, we find all the (mijmji), $i\neq j$. There are n(n-1) of them. Let rij=(mijmji)^i/2, $I\neq j$, and ($\prod$r1j)^1/(ni-1), ($\prod$r2j)^1/(n-1), ... ($\prod$rnj)^1/(n-1),I.e.the geometric mean of all the rij"s having the same i. Choose the smallest and largest of such numbers and call them g_s and g_l. Assume g_l is the largest. Let l_{sl}=(g_s/g_l)^1/2. Then u_s=l_{sl}u and u_l=u^2/u_s, and all the other u_k"s are u.

We went through all of these details in order to justify how we arrived at u_s and u_l. A much easier way to calculate them is: u_s=($\prod$mskmks/$\prod$mlkmkl)^1/(4(n-1))u and u_l=u^2/u_s, $k\neq s$,l. These new values for the u's bring us into the neighborhood of the best matrix. Once u_s is given, we vary it by a small amount. If we decrease it, which means we increase u_l and get a smaller λ, we continue to decrease it until λ stops decreasing and starts to increase. If it gets larger right away, then we increase u_s by degrees until λ stops getting smaller.

Once his happens, we keep u_s and u_l fixed and do the same thing with the next two extreme values. If n is even the process eventually ends. If n is odd, there is a u_k left over. We leave it as u.

The same algorithm can be applied to any mold M as defined in [2], p.19. Thus we have a way of finding the best matrix for any Pop(u) from any mold M.

Example

In [2], p.38 a local model was given which depended on the worst vector Pw(624.3 476.74 280.02 210.39), and the k's:
k1=.1723 k2=.14995 k3=.2001 k4=.2401. From these and si=ki/(1-ki) we get s1=.20817 s2=.1764 s3=.25016 s4=.31596 and the model

$$M=\begin{bmatrix} 1 & .2726 & .4641 & .6177 \\ .1347 & 1 & .3003 & .3997 \\ .1122 & .1469 & 1 & .33295 \\ .1065 & .1394 & .2374 & 1 \end{bmatrix}$$

From this , we obtain

$$Ae(.3)=\begin{bmatrix} .7 & .08178 & .13923 & .1853 \\ .04041 & .7 & .09009 & .11991 \\ .03366 & .04407 & .7 & .099885 \\ .03195 & .04182 & .07122 & .7 \end{bmatrix}$$

and from this matrix, λ_e=.9113433 .

We next find matrix Am. For that we need to calculate $\delta i=(1-\sum kj,\ j\neq i)(1-ki)$. Thus δ1=.33923 δ2=.32939 δ3=.35008 δ4=.36297, $\delta=(\prod \delta i)$^1/4, and ui=.3/$\delta\delta$iTherefore u1=.29482 u2=.28627 u5=.30425 u4=.31544, and

$$Am=\begin{bmatrix} .70513 & .080368 & .136826 & .1821103 \\ .038561 & .71373 & ,08596 & .114422 \\ .034137 & .044694 & .69575 & .1013 \\ .033595 & .043973 & .074886 & .6845564 \end{bmatrix}$$

which yields λ_m=.91109333. Thus we have Am<Ae in this case.It is quite possible that it is true in all cases except when n=2, where they are equal.

We next move on to the minimax algorithm. For that , we shall need m12m21m13m31m14m41, m21m12m23m32m24m42, etc.This gives us 1.2513x10^-4, 9.025422x10^-5, 1.99707x10^-4, 2.958322x10^-4.
Thus (9.025422/29.58322)^1/12).3=.2717. Letting u1=,3 u2=.2717 u3=.3 u4=3312477, we get λ=.911063519,which is already better than Am.

Continuing with the algorithm, we get the smallest eigenvalue keeping the first and third coordinate fixed at ,3.This is u1=.3 u2=.275 u3=.3 u4=.
327272
and λ=.9110593859. Keeping those coordinates fixed and varying the other two leads us to our best matrix

A*(..292 .275 .308219 .3272727)=

$$\begin{bmatrix} .708 & .0795992 & .1355172 & .1803684 \\ .0370425 & .725 & .0825835 & .1099175 \\ .034582 & .0452774 & .6917808 & .1026216 \\ .034854 & .0456218 & .0776945 & .67267267 \end{bmatrix}$$

In this case, λ=.911029867.

Reference

[1] Barsky,M. (1986) " A Matrix Method For Decision Making and Forecasting", Penn State U., Ogontz Campus, Abington, Pa.

[2] Barsky, M. (2019) "The Mathematics of Continuously changing Objects", Princeton, N,J.