A Matrix Method for Decision Making and Forecasting

Marvin Barsky

ABSTRACT

A new application of essentially non-negative matrices to business decisions is presented, where the matrix is connected to the business activity and each iteration represents the next time period. Thus, it can be used for both long range and short range forecasting and planning depending upon the length of each time interval.

Once again the Perron Frobenius theorem plays an important role here as it does throughout the theory of non-negative matrices.

1. Introduction

In this paper, a new method for decision making and forecasting is presented, the origins of which can be traced back to Cesàro summability and repeated convergence classes (see [1] and [2]).

From the past history of a firm, its size and near future expectations, a unique positive matrix can be constructed that represents the method of operation of that firm, *i.e.* how various units operate in conjunction with one another, how earnings are distributed among its units, its short term and long term trends, etc. The method has certain attractive features:

1. Since it represents the activity of the firm, it is less subject to randomness than if past history were used solely for predictive purposes.
2. The method is an extension of exponential smoothing which is an accepted technique for business forecasting.

3. The inputs that give rise to the matrix can be readily obtained from the size of the firm and its past history by using well known methods.
4. Since the theory of positive matrices is a rich theory that continues to grow, many of its non-trivial results can be reinterpreted back into the decision-making process.
5. The actual number of data required to contruct the unique $n \times n$ matrix is $5n$. Thus, it does not grow exponentially as dimensions are increased.
6. It can be used for short term or long term planning and for small businesses as well as large corporations.
7. The matrix gives rise to a forecasting mechanism, for by matrix multiplication on the present incomes of its units, it represents the continued activity of the firm, each iteration corresponding to the next time period.

There are five vectors that hold the required information about the firm needed to give rise to a unique positive matrix.

(i) The goal $G = (g_1, g_2, \ldots, g_n)$.

We assume that the firm consists of n units each bringing in "income." (We need not bother here with whether income is revenue or profit. For the purpose of constructing the matrix, we shall simply call these numbers "income".)

The goal values give us the size of the business. These values represent the maximum possible income that any unit can expect, given that there is no significant expansion occurring which would change the nature of the business and lead to a different matrix. Rather, there are so many hours in a day, so many employees that are working, *etc.*, and hence, given a certain sized business, there certainly is un upper limit of what can be expected.

For a healthy business, we assume that the present income values are not near the goal, meaning that there is room for improvement each time period without major expansions. As the goal is approached, income levels off with goal values being the limiting values. It is at such a time when expanding one's business should be considered. Thus, we assume present incomes are not near the goal and we can expect higher incomes each time period.

(ii) The present income P_S.

This is the amount actually made in each unit the last time period. We assume that nothing unusual happened then that would make these amounts not representative of the business incomes for that period.

To obtain the final three vectors

(iii) AP_S

(iv) P_B

(v) P_W

we need three projections of next time period's income, the most likely AP_S, along with the best P_B and worst P_W scenarios. We expect that AP_S will be greater than P_S, since, for a business to be in good shape, we must assume that it is growing. Since P_B is the best case scenario, it is obviously larger than AP_S, and P_W is less than AP_S. However, since this time period's actual income is P_S, the worst case scenario certainly would be less than what actually was made this time period, and therefore it is also less than P_S. Thus

$$P_W < P_S < AP_S < P_B < G.$$

The size of the interval between P_W and P_B is really an indication of how volatile the incomes are. If there can be large fluctuations of income from time period to time period, the interval will be large. If P_W and P_B are somewhat close together, it means that one would not expect that to happen.

What is also implicit in the interval P_W, P_B is that, no matter what income is actually made next time period, so long as it is between P_W and P_B, business can proceed as usual. In other words, the interval must not contain any points where the business will fail or where major revisions are necessary.

Certainly one can use linear regression and confidence intervals to come up with AP_S, P_B, and P_W along with a host of other well known techniques for short term projection.

2. Properties of the Matrix

Let $G = (g_1, g_2, \ldots, g_n)$ be an n-tuple of real numbers and **X** the set of all points $X = (x_1, x_2, \ldots, x_n)$ such that $x_i < g_i$, $i = 1, 2, \ldots, n$, $x_i \in R$.

Definition *The points will be said to be in standardized form, it we assign new coordinate to each point in* X *and to* G *as follows:*

$X = (x_1, x_2, \ldots, x_n)$ will be assigned coordinates

$(g_1 - x_1, g_2 - x_2, \ldots, g_n - x_n)$, and $G = (0, 0, \ldots, 0)$.

Thus, in standardized form, the goal G will be the origin and $\mathbf{X}$ *will be the set of all n-tuples of real numbers with positive entries.*

We return to pre-standardized form for the moment. In the one-dimensional case, $P_s = (x), G = (g)$, and if we are in a growth period, the matrix $A = (\lambda)$, where $0 < \lambda < 1$.

Then $P_s = (g - x)$ in standardized form, and $AP_s = (\lambda(g - x))$. Returning to the pre-standardized form, *i.e.* $G - AP_s$, $AP_s = ((1-\lambda)g + \lambda x)$ and we see that λ is the smoothing factor for exponential smoothing using the present state P_s and the goal G.

N iterations of A yields

$A^N P_s = ((1-\lambda^N)g + \lambda^N x)$, so that the smoothing factor changes to successive powers of λ. Obviously, as $N \to \infty$, $A^N P_s \to (g)$.

The matrix will be constructed from and applied to points in standardized form. From now on all points $X = (x_1, x_2, \ldots, x_n)$ will refer to its standardized coordinates.

Definition. *We say* $X \le Y$ *if* $x_i \le y_i$, $i = 1, 2, \ldots, n$, $X < Y$ *if* $X \le Y$ *and* $X \ne Y$, *and* $X << Y$ *if* $x_i < y_i$, $i = 1, 2, \ldots, n$.

It turns out that any matrix constructed from G, P_s, AP_s, P_W and P_B representing a business in a growth period has positive entries. For that reason, we shall consider matrices with positive or non-negative entries.

Theorem 1. *If* A *has non-negative entries, then* A *is order-preserving, i.e.* if $X, Y \in \mathbf{X}$ *and* $X \le Y$, *then* $AX \le AY$.

Proof. If $X \le Y$, then $Y - X \ge 0$. Hence $A(Y - X) \ge 0$ since the entries of A are non-negative. Thus $AY \ge AX$.

Theorem 2. *Let* A *be an* $n \times n$ *matrix with non-negative entries and suppose* $AX << X$ *for some* X, $AX \in \mathbf{X}$. *Then*

(i) *the diagonal elements* a_{ii} *of* A *are such that* $0 \le a_{ii} < 1$, $i=1,2,\ldots,n$.

(ii) *the co-factors* U_{ij} *of* $I-A$, I *the identity matrix, are all non-negative and* $0 < U_{ii} < 1$.

(iii) $0 < \chi(1) < 1$ *where* $\chi(x)$ *is the characteristic polynomial of* A, *i.e.* $\chi(x) = |xI - A|$.

Proof. The conditions of the theorem imply that $I-A$ is a non-singular M matrix (see Burman, Plemmons [4], p.136, I_{28}). Thus $1-a_{ii} > 0$, $U_{ii} > 0$ and $\chi(1) > 0$ since all the principal minors of $I-A$ are positive.

Furthermore $(I-A)^{-1} > 0$ implying that $U_{ij} \ge 0$. $U_{ii} < 1$ and $\chi(1) < 1$ are trivial consequences. This proves the theorem. ■

In what follows, we shall have occasion to use the expression $1-\chi(1)$ which will be denoted by p. Thus (iii) is equivalent to

(iii') $0 < p < 1$

Definition. *The set of all vectors* $X \in \mathbf{X}$ *such that* $AX << X$ *will be called the improvement region of* A *and denoted by* $\mathbf{I}_A$. *Let*

$$V_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ \cdot \\ \cdot \\ U_{in} \end{bmatrix}, \qquad i = 1,2,\ldots,n.$$

From Theorem 2, the V_i *have all non-negative entries. The next theorem shows that* $\{V_1, V_2, \ldots, V_n\}$ *forms the boundary vectors of a conical region whose interior is the improvement region* $\mathbf{I}_A$.

Theorem 3. $X \in \mathbf{I}_A$ *if and only if* $X = \sum_{i=1}^{n} c_i V_i$ *where* $c_i > 0$, $i = 1,2,\ldots,n$.

Proof. Since the column vectors of Adj$(I-A)$ are the V_i's we have

$$(I-A)\text{Adj}(I-A) = \chi(1)I$$

and it follows that $AV_i = V_i - \chi(1)I_i$ where I_i is the i'th column of the identity matrix.

Since the V_i are independent they form a basis for R^n. Thus, for any $X \in R^n$ we can write

$$X = \sum_{i=1}^{n} c_i V_i$$

and

$$AX = \sum_{i=1}^{n} c_i A V_i = \sum_{i=1}^{n} c_i (V_i - \chi(1) I_i)$$

$$= X - \chi(1) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Hence $X \in \mathrm{I}_A$ if and only if $c_i > 0,\ i = 1, 2, \ldots, n$. This proves the theorem. ■

Theorem 4. *Let A be an $n \times n$ matrix with non-negative entries such that $\mathrm{I}_A \neq \phi$. For any $X \in \mathrm{I}_A$,*

$$A^n X < pX.$$

Proof. We will show that $A^n V_i < pV$ for $i = 1, 2, \ldots, n$ where the V_i's are the vectors bounding I_A. Since any $X \in \mathrm{I}_A$ is sucl that

$$X = \sum_{i=1}^{n} c_i V_i, \qquad c_1, c_2, \ldots, c_n > 0$$

the theorem will follow.

From the proof of Theorem 3 we saw that

$$AV_i = V_i - \chi(1)I_i.$$

Claim 1. $U_{ii} - \chi(1) < [1-\chi(1)]U_{ii} = pU_{ii}$

Proof. $[1-\chi(1)]U_{ii} - [U_{ii} - \chi(1)] = \chi(1)[1-U_{ii}] > 0$ since $\chi(1) > 0$ and $U_{ii} < 1$.

But $U_{ii} - \chi(1)$ is the i'th entry of AV_i. Thus that entry has already been diminished by the factor p after only one application of A. Now

$$A^n V_i = V_i - \chi(1)[I_i + AI_i + \ldots + A^{n-1}I_i] = [1-\chi(1)]V_i - \chi(1)[I_i \in AI_i + \ldots. + A^{n-1}I_i - V_i]$$

Claim 2. $I_i + AI_i + \ldots + A^{n-1}I_i - V_i \geq 0$. Now

$$AI_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ a_{ni} \end{bmatrix}$$

$$A^2 I_i = \begin{bmatrix} \sum_{j_1=1}^{n} a_{ij_1} a_{j_1 i} \\ \cdot \\ \cdot \\ \sum_{j_1=1}^{n} a_{nj_1} a_{j_1 i} \end{bmatrix}$$

$$\vdots$$

$$A^k I_i = \begin{bmatrix} \sum_{j_{k-1}} \sum_{j_{k-2}} \cdots \sum_{j_1} a_{ij_{k-1}} a_{j_{k-1} j_{k-2}} \cdots a_{j_2 j_1} a_{j_1 i} \\ \cdot \\ \cdot \\ \sum_{j_{k-1}} \sum_{j_{k-2}} \cdots \sum_{j_1} a_{nj_{k-1}} a_{j_{k-1}} a_{j_{k-2}} \cdots a_{j_2 j_1} a_{j_1 i} \end{bmatrix}$$

Notice that each entry of $I_i + AI_i + \ldots + A^{n-1}I_i$ is made up of sums of products of entries of A. We shall refer to a product of k entries of A as a k factor term. Thus, AI_i consists of a sum of 1 factor, A^2I_i, a sum of 2 factors, . . . , $A^{n-1}I_i$, a sum of $(n-1)$ factors. Also

$$V_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ \cdot \\ \cdot \\ U_{in} \end{bmatrix}$$

Consider a given entry U_{ij} where $i \neq j$. Since U_{ij} is the ij'th co-factor of $I - A$, if we expand it, the resulting expression will consist of sums and differences of k factors for $k = 1,2, \ldots, n-1$. Subtracting from $(I_i)+A\,(I_i)+\ldots+A^{n-1}(I_i)$, one can note that every k factor term for $k < n-1$ that ends up as a negative term can be canceled with the identical positive term of $(I_i)+A\,(I_i)+\ldots+A^{n-1}(I_i)$. As for the $(n-1)$ factor terms, either they can also be canceled in the same way, or they are of the form

$$-\prod_{j=1}^{n-1} a_{ij}\, a_{kk}$$

for some $k = 1,2, \ldots, n$ in which there is a positive term $\prod_{j=1}^{n-1} a_{ij}$ also from U_{ij}. Combining these two terms, we get

$$\prod_{j=1}^{n-2} a_{ij}(1-a_{kk}) \geq 0$$

since $a_{kk} < 1$. Thus,

$$I_i + AI_i + \ldots + A^{n-1}I_i - V_i > 0.$$

This proves the claim. Since

$$A^n V_i = pV_i - \chi(1)[I_i + AI_i + \ldots + A^{n-1}I_i - V_i],$$

it follows that

$$A^n V_i < pV_i$$

This proves the theorem. ■

Corollary 4.1. *Given the conditions of Theorem 4,*

$$A^{kn} X < p^k X$$

Proof. Since A is order-preserving

$$A^{2n} X = A^n (A^n X) < A^n (pX) = pA^n X < p^2 X.$$

Thus, $A^{kn}X < p^k X$.

From Theorem 4, we see that p represents a guaranteed amount of improvement for any vector in I_A after n applications of A. For that reason p will be referred to as the *rate of improvement* of A. Note that any particular vector $X \in I_A$ may improve at a faster rate than p, but none can improve more slowly. p sets the minimal amount of improvement for any factor of any vector in I_A after n applications of A.

Definition. *A square non-negative matrix A is said to be primitive if there exists a positive integer k such that* $A^k >> 0$.

Theorem 5. *If A is a primitive matrix such that* $I_A \neq \phi$, *then*

(i) $\lambda < 1$ *where* λ *is the Perron Frobenius eigenvalue*

(ii) *for any* $X \in \mathbf{X}$, $\lim_{N \to \infty} \frac{x_i^{(N)}}{x_i^{(N-1)}} = \lambda$, *where* $x_i^{(N)}$ *is the i'th row of the column vector* $A^N X$.

(iii) *if* $X \in I_A$, *then*

$$\min_{1 \le i \le n} \frac{x_i^{(N)}}{x_i^{(N-1)}} \text{ monotonically increases to } \lambda$$

and

$$\max_{1 \le i \le n} \frac{x_i^{(N)}}{x_i^{(N-1)}} \text{ monotonically decreases to } \lambda$$

as $N \to \infty$.

Proof. Since $I - A$ is a non-singular M matrix (i) is an immediate consequence of the Perron Frobenius theorem (see [14], pp. 3-4). It also follows from that theorem that there is an eigenvector of λ with all positive entries. We shall denote such an eigenvector by Γ_n. Assume at first that all the eigenvalues of A are distinct. Then a set of corresponding eigenvectors Γ_i form a basis of R^n. Thus any $X \in R^n$ can be written as $X = \sum_{i=1}^{n} c_i \Gamma_i$. Since $\lambda = \lambda_n$ is a real number larger than the modulus of any other eigenvalue,

$$\lim_{N \to \infty} \frac{1}{\lambda^N} A^N X = \begin{cases} c_n \Gamma_n & \text{if } c_n \neq 0 \\ 0 & \text{if } c_n = 0 \end{cases}$$

Claim 1. If $X \in \mathbf{X}$ then $c_n \neq 0$

Proof. We can find an $\varepsilon > 0$ so that $\varepsilon\Gamma_n < X$ for any $X \in \mathbf{X}$. But $1/\lambda^N A^N(\varepsilon\Gamma_n) \leq 1/\lambda^N A^N X$. The left-hand side tends to $\varepsilon\Gamma_n$ as $N \to \infty$. Hence $\lim_{n \to \infty} 1/\lambda^N A^N X \geq \varepsilon\Gamma_n > 0$. Therefore $c_n \neq 0$. In fact $c_n > 0$. Let $\Gamma_j = \begin{bmatrix} \delta_{j1} \\ \cdot \\ \cdot \\ \delta_{jn} \end{bmatrix}$. Then

$$\frac{x_i^{(N)}}{x_i^{(N-1)}} = \frac{\sum_{j=1}^{n} \lambda_j^N c_j \delta_{ji}}{\sum_{j=1}^{n} \lambda_j^{N-1} c_j \delta_{ji}} = \frac{\lambda_n^N}{\lambda_n^{N-1}} \frac{\sum_{j=1}^{N} \left(\frac{\lambda_j}{\lambda_n}\right)^N c_j \delta_{ji}}{\sum_{j=1}^{n} \left(\frac{\lambda_j}{\lambda_n}\right)^{N-1} c_j \delta_{ji}} \to \lambda_n = \lambda \text{ as } N \to \infty.$$

If not all the eigenvalues are distinct then we can vary the matrix so that they are, and by continuity we obtain the same result.

Lemma 5.1. *Suppose $X \in \mathbf{I}_A$. If*

$$AX = A \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 x_1 \\ p_2 x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix}$$

and

$$A^2 X = \begin{bmatrix} q_1 p_1 x_1 \\ q_2 p_2 x_2 \\ \cdot \\ \cdot \\ q_n p_n x_n \end{bmatrix},$$

then the q's lie between the p's.

Proof.

Suppose $p^* = \max_{1 \leq i \leq n} \{p_i\}$. then $\begin{bmatrix} p_1 x_1 \\ p_2 x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix} < p^* \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$.

Since A is order-preserving

$$A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix} \le p^* A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix}. \text{ Thus } \begin{bmatrix} q_1p_1x_1 \\ q_2p_2x_2 \\ \cdot \\ \cdot \\ q_n p_n x_n \end{bmatrix} \le \begin{bmatrix} p^*p_1x_1 \\ p^*p_2x_2 \\ \cdot \\ \cdot \\ p^* p_n x_n \end{bmatrix}.$$

Thus $q_i \le p^*$ for $1 \le i \le n$.

Similarly, if $p_* = \min_{1 \le i \le n} \{p_i\}$ then

$$A\begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix} \ge p_* \begin{bmatrix} p_1x_1 \\ p_2x_2 \\ \cdot \\ \cdot \\ p_n x_n \end{bmatrix} \text{ and } q_i \ge p_*, \ i = 1,2,\ldots,n.$$

Thus, (iii) follows from this lemma and the first part of the proof. This proves the theorem. ■.

3. Improvement Regions Induced by Two Points

We now consider that we are given two points P_W and P_B representing the worst and best scenarios that can be expected at the end of the next time period and we consider the set of points P such that $P_B < P < P_W$. This set gives rise to a unique cone, the smallest cone that contains that set.

This can easily be illustrated in two dimensions (see Figure 1). The set of points between $P_B(b_1,b_2)$ and $P_W(w_1,w_2)$ is the shaded region. All cones representing improvement regions are bounded by two rays starting at the origin. Obviously there is a smallest cone containing the set whose bounding rays pass through (b_1,w_2) and (w_1,b_2) respectively.

This improvement region will be called the improvement region induced by P_W and P_B.

Since the set of points between P_W and P_B represents the area where business can proceed as usual, which inevitably implies an expectation of growth for the future, and since the induced improvement region consists of only non-negative multiples of vectors in that set, we can associate the business activity with only those matrices that have that induced improvement region.

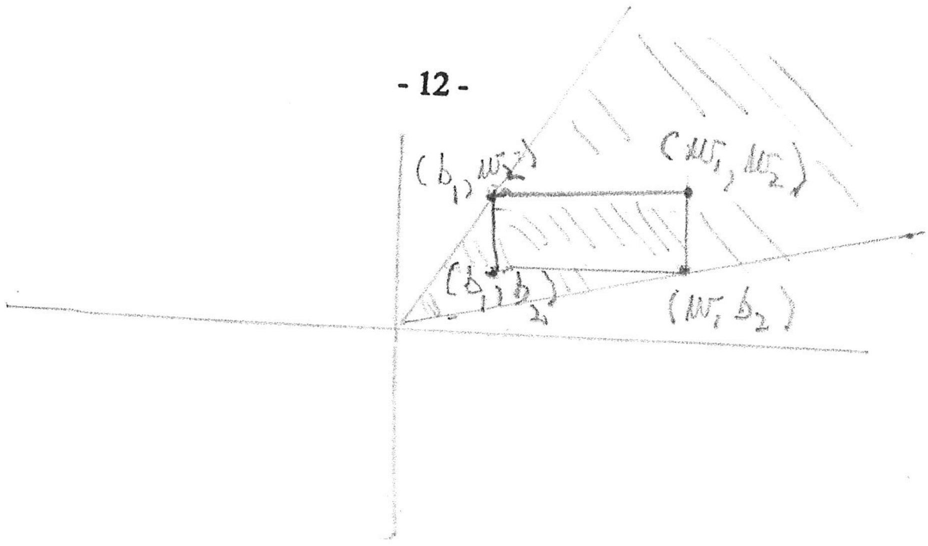

Figure 1.

Turning now to the n-dimensional case we shall construct a formula that yields all such matrices.

At first, suppose we are given points

$$P_W = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \text{ and } P_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \qquad 0 < \alpha_i < 1.$$

We first construct a matrix A^* that has an improvement region induced by P_W and P_B. Let

$$A^* = \begin{bmatrix} k_1 & k_1 & \dots & k_1 \\ k_2 & k_2 & \dots & k_2 \\ \cdot & & & \\ \cdot & & & \\ k_n & k_n & \dots & k_n \end{bmatrix} \qquad 0 < k_i < \frac{1}{n} \quad i = 1,2,\dots,n.$$

Then

$$A^* \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

and A^* has an improvement regions $I_{A^*} \neq \phi$. We shall show that for special values of the k_i, I_{A^*} is induced by P_W and P_B.

Vertices of the region between P_W and P_B are of the form

$$P_j = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \alpha_i \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad j = 1,2,\ldots,n.$$

Furthermore, since every point in Γ_{A^*} is of the form

$$\sum_{i=1}^{n} c_i V_i^*, \qquad c_i \geq 0, \qquad V_i^* = \begin{bmatrix} U_{i1}^* \\ \cdot \\ \cdot \\ U_{in}^* \end{bmatrix}$$

where U_{ij}^* are the cofactors of $I - A^*$; and, since each P_j lies on the hyperplane with basis $V_1^*, V_2^*, \ldots, V_{j-1}^*, V_{j+1}^*, \ldots, V_n^*$, we have

$$P_j = \sum_{i=1}^{n} c_{ji} V_i^* \qquad \text{where } c_{jj} = 0. \tag{6}$$

We calculate V_i^* from the matrix $I - A^*$. Thus

$$I - A^* = \begin{bmatrix} 1-k_1 & -k_1 & \ldots & -k_1 \\ -k_2 & 1-k_2 & \ldots & -k_2 \\ \cdot & & & \\ \cdot & & & \\ -k_n & -k_n & \ldots & 1-k_n \end{bmatrix}$$

From this matrix it is readily verified that

$$V_i^* = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ k_{i-1} \\ 1 - \sum_{j \neq i} k_j \\ k_{i+1} \\ k_n \end{bmatrix}$$

Therefore (6) leads to the system of $n-1$ linear equations in unknowns $c_{ji},\ i = 1,2,\ldots,n,\ i \neq j$.

Solving this system, we obtain by Cramer's rule

$$c_{ji} = \frac{1 - k_j - (n-1)k_i}{\left[1 - \sum_{l=1}^{n} k_l\right](1-k_j)}. \tag{7}$$

Note that because of the restriction on the k's, the determinant of the coefficients is greater than 0. Using the j'th equation $\sum_{i \neq j} c_{ji} k_j = \alpha_j$ and substituting for the c_{ji}'s in (7) we obtain

$$\sum_{i \neq j} c_{ji} k_j = \frac{\sum_{i \neq j} [(1-k_j)-(n-1)k_i]}{\left[1-\sum_{i=1}^{M} k_i\right](1-k_j)} = \frac{(n-1)k_j}{1-k_j} = \alpha_j .$$

Thus, solving for k_j we get

$$k_j = \frac{\alpha_j}{n-1+\alpha_j} \qquad j=1,2,\ldots,n. \tag{8}$$

NOTE: The i'th coordinate α_i of P_i remains invariant under A^* *i.e.*, $A^* P_i$ has i'th coordinate α_i.

To see this, it is only necessary to observe that the i'th coordinate of $A^* P_i$ is

$$(n-1)k_i + k_i \alpha_i = k_i(n-1+\alpha_i) = \frac{\alpha_i}{n-1+\alpha_i}(n-1+\alpha_i) = \alpha_i .$$

Moving on to the general case, suppose

$$P_W = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \text{ and } P_B = \begin{bmatrix} \alpha_1 w_1 \\ \alpha_2 w_2 \\ \cdot \\ \cdot \\ \alpha_n w_n \end{bmatrix} \qquad 0 < \alpha_i < 1, \; i=1,2,\ldots,n.$$

We can reduce this to the previous case by using the matrix

$$A^* = \begin{bmatrix} \frac{w_1}{w_1}k_1 & \frac{w_1}{w_2}k_1 & \cdots & \frac{w_1}{w_n}k_1 \\ \frac{w_2}{w_1}k_1 & \frac{w_2}{w_2}k_2 & \cdots & \frac{w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{w_n}{w_1}k_1 & \frac{w_n}{w_2}k_2 & \cdots & \frac{w_n}{w_n}k_n \end{bmatrix}.$$

Going through an analogous argument, we arrive at the same expression (8) for the k_j's.

Now

$$I-A^{*}=\begin{bmatrix} 1-\frac{w_1}{w_1}k_1 & \frac{-w_1}{w_2}k_1 & \cdots & \frac{-w_1}{w_n}k_1 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}k_1 & & \cdots & 1-\frac{w_n}{w_n}k_n \end{bmatrix}.$$

If we multiply each element in the i'th row of U^{*} by a positive number l_i, we get a scalar multiple of V_i in the same direction. Thus, this set of vectors generates the same improvement region.

Let

$$U=\begin{bmatrix} l_1(1-k_1) & l_1\frac{w_1}{w_2}k_1 & \cdots & -l_1\frac{w_1}{w_n}k_1 \\ -l_2\frac{w_2}{w_1}k_2 & l_2(1-k_2) & \cdots & -l_2\frac{w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ -l_n\frac{w_n}{w_1}k_1 & & \cdots & l_n(1-k_n) \end{bmatrix}$$

If we now define A so that $U=I-A$ then $A=I-(I-A)$ and therefore

$$A=\begin{bmatrix} 1-l_1(1-k_1) & \frac{w_1}{w_2}l_1k_1 & \cdots & \frac{w_1}{w_n}l_1k_1 \\ \frac{w_2}{w_1}l_2k_2 & 1-l_2(1-k_2) & \cdots & \frac{w_2}{w_n}l_2k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{w_n}{w_1}l_nk_n & \frac{w_n}{w_2}l_nk_n & \cdots & 1-l_n(1-k_n) \end{bmatrix} \tag{9}$$

The l's are not any positive numbers, but since the main diagonal elements of A must be numbers between 0 and 1, we must have

$$0\le 1-l_i(1-k_i)<1$$

or

$$l_i\le\frac{1}{1-k_i}\qquad i=1,2,\ldots,n. \tag{10}$$

Now consider the matrix

$$B = \begin{bmatrix} (1-k_1) & \frac{-w_1}{w_2}k_1 & \cdots & \frac{-w_1}{w_n}k_1 \\ \frac{-w_2}{w_1}k_2 & (1-k_2) & \cdots & \frac{-w_2}{w_n}k_2 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}k_n & \frac{-w_n}{w_2}k_n & \cdots & (1-k_n) \end{bmatrix}$$

It can be readily shown that

$$\det B = 1 - \sum_{i=1}^{n} k_i$$

But

$$I - A = \begin{bmatrix} l_1(1-k_1) & \frac{-w_1}{w_2}l_1k_1 & \cdots & \frac{-w_1}{w_n}l_1k_1 \\ \cdot & & & \\ \cdot & & & \\ \frac{-w_n}{w_1}l_nk_n & & \cdots & l_n(1-k_n) \end{bmatrix}$$

and therefore

$$\chi(1) = \det(I - A) = \prod_{i=1}^{n} l_i (\det B) = \prod_{i=1}^{n} l_i [1 - \sum_{i=1}^{n} k_i]$$

Thus

$$p = 1 - \prod_{i=1}^{n} l_i \left[1 - \sum_{i=1}^{n} k_i\right]. \qquad (11)$$

Since $l_i \le \frac{1}{1-k_i}$, we have

$$\prod_{i=1}^{n} l_i \le \prod_{i=1}^{n} \frac{1}{1-k_i}.$$

Hence

$$p \ge 1 - \prod_{i=1}^{n} \left[\frac{1}{1-k_i}\right] + \frac{\sum_{i=1}^{n} k_i}{\prod_{i=1}^{n}(1-k_i)}$$

or

$$\frac{\prod_{i=1}^{n}(1-k_i) + \sum_{i=1}^{n} k_i - 1}{\prod_{i=1}^{n}(1-k_i)} \le p < 1. \qquad (12)$$

We now have the necessary formulae to obtain a unique matrix from the inputs G, P_W, P_B, P_S and AP_S. First we use G to standardize the coordinates. In standardized form, if

$$P_W = (w_1, w_2, \ldots, w_n),\; P_B = (b_1, b_2, \ldots, b_n),\; P_S = (x_1, x_2, \ldots, x_n),\; AP_S = (y_1, y_2, \ldots, y_n),$$

we find $\alpha_i = \frac{b_i}{w_i}$, $i = 1, 2, \ldots, n$. We then use (8) to calculate the k's, *i.e.* $k_i = \frac{\alpha_i}{n-1+\alpha_i}$. Since

$$A \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix},$$

we can solve for the l's. Letting $s = \sum_{i=1}^{n} \frac{x_i}{w_i}$, we find that

$$l_i = \frac{y_i - x_i}{w_i k_i s - x_i}. \tag{13}$$

Hence, substituting these values in (9), we obtain our matrix A. Once we have the matrix, it can be used to forecast future income by applying it successively to the present state P_S, each application being a forecast of the next time period.

If there are several alternative business ventures to choose from, we can set up a matrix for each one and by forecasting future results, we can compare how each one progresses. Note that our forecasts do not simply yield a total value for each time period, but rather we can trace how each factor does over any desired number of time periods.

If the alternatives all have the same goal values then we can compare the characteristics of the matrices themselves. The matrix with the smallest λ will eventually give the best results since, by Theorem 5, after enough iterations, the next iteration will be approximately the same as multiplying the resulting vector by λ. Hence, the matrix with the smallest λ will approach the goal more rapidly in each factor. On the other hand, even though that must happen eventually, it might require an unreasonable number of applications of the matrix and therefore may not be the matrix of choice for a particular real situation.

It may be that the main concern of a business is the requirement that every factor increase its earnings by a certain percentage within a certain time frame. In that case we would want the matrix with the smallest p. Furthermore, once we have our best and worst scenario, (12) gives us our smallest possible value of p regardless of what our forecast for AP_S is. If this value is not suitable, then it indicates that the business activity itself should be restructured.

4. Matrices with Same Improvement Region and Rate of Improvement

In this section we consider all matrices that have the same improvement region I_A and rate of improvement p. This set forms an equivalence class denoted by $\mathcal{E}(\mathrm{I}_A, p)$. Fixing I_A leads to fixing P_W and the k's in (9). In addition, fixing p fixes $\prod_{i=1}^{n} l_i$. This comes from (11), *i.e.*, $p = 1 - \prod_{i=1}^{n} l_i \left[1 - \sum_{i=1}^{n} k_i\right]$.

Thus, if we are given P_W and P_B which fixes I_A and we fix p, then only the l's in the matrix (9) can vary subject to the restriction $\prod_{i=1}^{n} l_i$ is fixed. But from (9) we see that for any factor x_i of P_S, the smaller we make l_i in our matrix, the less influence there is from the other factors on future values of the i'th coordinate. The larger we make l_i, the more the other factors influence the future values of this coordinate. Since $\prod_{i=1}^{n} l_i$ is fixed, each change in the values of the l's represent a redistribution of time and money put into each factor.

Thus we make the following assumption:

The total cost and the total time for running the business as given by G, P_W, P_B, P_S, and AP_S remains fixed for all matrices in $\mathcal{E}(\mathrm{I}_A, p)$ during the first application of A. Each matrix in the class represents a different distribution of those times and funds over all of the factors, subject to the restrictions on the l's, *i.e.*

$$0 < l_i < \frac{1}{1-k_i}, \qquad \prod_{i=1}^{n} l_i = \frac{1-p}{1-\sum_{i=1}^{n} k_i}, \qquad \text{see (11).}$$

What is significant about the interpretation is that matrices in $\mathcal{E}(\mathrm{I}_A, p)$ have different λ's for the most part. By finding the matrix with the smallest λ, the "best matrix," we will be finding that distribution of our funds and our time that will lead eventually to our best returns. It will allow us to streamline our business to obtain optimal results without increasing our costs. On the other hand, matrices in $\mathcal{E}$ with the largest λ occur when some of the $l_i = \frac{1}{1-k_i}$, the endpoint of their allowed interval. If the i"th source of income is capable of greatly increasing as projected by its goal values compared to the other sources, then letting $l_i = \frac{1}{1-k_i}$ puts the maximum permissible amount of time and money toward developing that source without causing the others to lose income, which could lead in some cases to a sizable increase of total income right from the start. But it will be done at the expense of keeping the other sources barely increasing at all. Eventually any other activity represented by a matrix in $\mathcal{E}$ will surpass it, but in some cases, by the time it does, either too many iterations are necessary or all values are almost at the goal, where minute differences are of no importance. Before then, however, this "worst" matrix is out-performing all others for many time periods which would certainly make it the most desirable matrix to choose. Thus, once we have the matrix A representing our business activity, if we wish to consider whether or not to change that activity without changing our total time and cost, it is well to consider the "best" and "worst" matrices. Through them, we will be able to decide what kind of change, if any, will be most appropriate for our business.

In order to help simplify the following results, we shall introduce some new variables.

Let $\eta_i = l_i(1-k_i)$, $i = 1,2,\ldots,n$ and $\eta = \left[\prod_{i=1}^{n} \eta_i\right]^{1/n}$. Note that η is the geometric mean of $\eta_1, \eta_2, \ldots, \eta_n$. We shall also let

$$s_0 = 1,\quad s_j = \frac{\sum_{i_1 < i_2 < \ldots < i_j} (1-k_{i_1}-k_{i_2}-\ldots-k_{i_j})\eta_{i_1}\eta_{i_2}\ldots\eta_{i_j}}{\prod_{l=1}^{j}(1-k_{i_l})},\qquad j = 1,2,3,\ldots,n. \tag{12}$$

Finally, let

$$\eta_i = \varepsilon_i \eta. \tag{13}$$

It is well known that

$$X(x) = \sum_{k=0}^{n} (-1)^k S_k x^{n-k} \tag{14}$$

where $S_k =$ sum of the principal minors of A of order k (see [10]), p. 198, for example).

Letting A be given by (9), we see that

$$S_j=\sum_{i=0}^{j-1}(-1)^{i+n}(n-i)s_i+(-1)^{i+n}s_j. \tag{15}$$

Replacing (15) into (14) and rearranging terms, we obtain

$$X(x)=\sum_{i=0}^{n}s_i(x-1)^{n-i}. \tag{16}$$

Theorem 6. *If all the k's have the same value, then the matrix with the smallest λ will be that unique matrix all of whose main diagonal entries are equal. In this case*

$$\lambda=1-\frac{1-nk}{1-k}\eta.$$

Proof. From (16) we write $X(x)=\sum_{i=0}^{n}s_i(x-1)^{n-i}$ where, under the assumption of equal k's, we can write

$$s_i=\frac{1-ik}{(1-k)^i}\left[\sum_{j_1<j_2<\ldots<j_i}\varepsilon_{j_1},\varepsilon_{j_2},\ldots,\varepsilon_{j_i}\right]\eta^i.$$

We keep x fixed between 0 and 1, and k fixed, and consider X as a function of $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n-1}$, since

$$\varepsilon_n=\frac{\eta^n}{\prod_{i=1}^{n-1}\varepsilon_i}.$$

We seek to find the critical points of X. Taking first order partials, we obtain after simplification

$$\frac{\partial X}{\partial\varepsilon_i}=(x-1)\left[\prod_{\substack{j=1\\ j\neq 1}}^{n-1}\varepsilon_j\varepsilon_i^2-1\right]\eta P_i(x), \tag{*}$$

where

$$P_i(x)=(x-1)^{n-2}+\frac{1-2k}{(1-k)^2}\left[\sum_{\substack{j=1\\ j\neq 1}}^{n-1}\varepsilon_i\right](x-1)^{n-3}\eta+\frac{1-3k}{(1-k)^3}\left[\sum_{\substack{j_1,j_2=1\\ j_1,j_2\neq i\\ j_1\neq j_2}}^{n-1}\varepsilon_{j_1}\varepsilon_{j_2}\right](x-1)^{n-4}\eta^2$$

$$+\ldots+\frac{1-(n-1)k}{(1-k)^{n-1}}\prod_{\substack{j=1\\ j\neq i}}^{n-1}\varepsilon_j\eta^{n-2}.$$

Setting $\frac{\delta X}{\delta \varepsilon_i}=0,\ i=1,2,\ldots,n-1$, we have either $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=1$ or, in order that all $P_i(x)=0$, we must have $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=\varepsilon$.

Lemma 6.1. *Let $X_\varepsilon(x)$ be the characteristic polynomial where $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_{n-1}=\varepsilon$. Then, if $x_\varepsilon=1-\frac{\varepsilon}{1-k}\eta$,*

$$X_\varepsilon(x)=(x-x_\varepsilon)^{n-2}\left\{(x-1)^2+\left[1-\frac{(n-1)k}{1-k}\varepsilon+\frac{1}{\varepsilon^{n-1}}\right]\eta(x-1)+\frac{1}{\varepsilon^{n-2}}\frac{1-nk}{(1-k)^2}\eta^2\right\}.$$

In particular, this implies that x_ε is a root of multiplicity $n-2$.

This can be proved by induction.

Corollary 6.1.

$$\lambda_\varepsilon=1-\left\{\frac{1}{2}\left[\frac{1-(n-1)k}{1-k}\varepsilon+\frac{1}{\varepsilon^{n-1}}\right]\eta-\frac{1}{2}\sqrt{\frac{1-(n-1)k}{(1-k)}\varepsilon+\left[\frac{1}{\varepsilon^{n-1}}\right]^2-\frac{4(1-nk)}{\varepsilon^{n-2}(1-k)^2}\eta}\right\}$$

This comes from using the quadratic formula on the expression in braces and selecting the largest root.

Corollary 6.2. *x_ε is a root of multiplicity $n-3$ of $\frac{dX_\varepsilon}{d\varepsilon}$.*

Lemma 6.2. *$\bar{x}_\varepsilon=1-\frac{1-(n-1)k}{(1-k)^2}\varepsilon\eta$ is a root of $\frac{dX_\varepsilon}{d\varepsilon}$.*

This can be shown by direct substitution. Since $P_i(x)$ is of order $n-2$, the only critical values of X are x_ε and $\bar{x}_\varepsilon$. Furthermore, $x_\varepsilon<\bar{x}_\varepsilon\le\lambda_\varepsilon$ as can readily be shown.

Lemma 6.3.

$$\left[\frac{1-P}{1-nk}\right]^{\frac{n-1}{n}}(1-k)^{n-1}\le\varepsilon_i\le\left[\frac{1-nk}{1-P}\right]^{1/n}\frac{1}{1-k}.$$

The proof comes about in a straightforward fashion from $l_i\le\frac{1}{1-k},\ i=1,2,\ldots,n$ and $P=1-\prod_{i=1}^{n}l_i[1-nk]$.

Now $X_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-2},\varepsilon_{n-i}} - X_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{n-1}}$ for some $i \neq n-1$

$$= \frac{(x-1)\left[\prod_{j=1}^{n-2}\varepsilon_j\varepsilon_{n-1}^2 - 1)(\varepsilon_{n-1}-\varepsilon_{n-1})\right]}{\prod_{j=1}^{n-2}\varepsilon_i\varepsilon_{n-1}^2} P_{n-1}(x).$$

Lemma 6.4. *Let* $\bar{x}_{\bar{\varepsilon}} = 1 - \frac{1-(n-1)k}{(1-k)^2}\bar{\varepsilon}\eta$. *Then*

$$P_{n-1}(\bar{x}_{\bar{\varepsilon}}) = \sum_{l=1}^{n-3} a_l \sum_{C_l} \prod_{i=1}^{l}(\varepsilon_{j_i} - \bar{\varepsilon}) + a_{n-2}\left[\prod_{j=1}^{n-2}\varepsilon_j - \bar{\varepsilon}^{n-2}\right],$$

where $\sum_{C_l}$ *is over all combinations of* ε_j *taken* l *at a time and where the* $a_l = a_l(k, \bar{\varepsilon}) > 0$.

This can be proved by induction on n.

Lemma 6.5. $X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$ *has a maximum at* $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{n-1} = 1$ *for all* x *such that* $x \geq 1 - \frac{1-(n-1)k}{(1-k)^2}\eta$. *Furthermore it is the only maximum in that range of* x.

Proof. First note that

$$\frac{\partial X}{\partial \varepsilon_i} = (x-1)\frac{\prod_{\substack{j=1 \\ j \neq 1}}^{n-1}(\varepsilon_j\varepsilon_i^2 - 1)\eta}{\prod_{j=1}^{n-1}\varepsilon_j\varepsilon_i^2} P_i(x),$$

where $P_i(x)$ is defined by (*). For $\varepsilon_i = 1$, $i = 1, 2, \dots, n-1$, we see that $\frac{\partial X}{\partial \varepsilon_i} = 0$ regardless of $P_i(x)$. If all the ε_i's are not 1, then $\frac{\partial X}{\partial \varepsilon_i} = 0$ only when all $P_i(x) = 0$ and that occurs only when $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{n-1} = \varepsilon$ and $x = 1 - \frac{1-(n-1)k}{(1-k)^2}\varepsilon\eta$. If $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{n-1} = \varepsilon$, then we denote $X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$ by X_ε. If $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_{n-2} = \varepsilon$, $\varepsilon_{n-1} = \sigma$, we denote $X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1})$ by $X_{\varepsilon,\sigma}$.

But, from Lemma 6.4, if $\varepsilon < 1$, then

$$X_\varepsilon(\bar{x}_\varepsilon) > X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma < \varepsilon$$

$$X_\varepsilon(\bar{x}_\varepsilon) < X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma > \varepsilon.$$

This means that $(\varepsilon, \varepsilon, \dots, \varepsilon, \bar{x}_\varepsilon)$ is a saddle point for $\varepsilon < 1$.

Likewise, for $\varepsilon > 1$,

$$X_\varepsilon(\bar{x}_\varepsilon) < X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma < \varepsilon$$

$$X_\varepsilon(\bar{x}_\varepsilon) > X_{\varepsilon,\sigma}(\bar{x}_\varepsilon) \quad \text{if } \sigma > \varepsilon.$$

Thus $(\varepsilon, \varepsilon, \ldots, \varepsilon, \bar{x}_\varepsilon)$ is a saddle point for all $\varepsilon \neq 1$. However, for $\varepsilon = 1$, $(1, 1, \ldots, 1, \bar{x}_1)$ is a maximum. Furthermore $X_1(\bar{x}_1) < 0$.

Since the ε_i's are bounded in a closed interval from Lemma 6.3, and since $f(X_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}) = \lambda_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}}$ is a continuous function, it attains its maximum and minimum in that interval. Since maximum λ is obtained on the boundaries of the ε_i as can readily be shown, we can see that $x_{1,1,\ldots,1}$ is the only polynomiāl that maximizes all of the others in some neighborhood around it and has a smaller λ than any of the others.

This completes the proof of Theorem 6.

When the k's are not all the same, the question arises whether the matrix, all of whose main diagonal elements are the same, is still the "best matrix", *i.e.* the one with smallest λ. The answer to this is "no" as will be shown.

Since all characteristic polynomials $X(x)$ in the equivalence class $\varepsilon(I_{A,P})$ pass though the point $(1, X(1))$ and have positive slopes at that point, one can ask for the polynomial whose slope is a minimum, since that curve will be above all the others, at least in a neighborhood of that point. In order to find this matrix, we first find the slope of $X(x)$ for $x = 1$, *i.e.*,

$$\left.\frac{\partial X}{\partial x}\right]_{x=1} = s_{n-1}.$$

Theorem 7. *The matrix whose slope at* $(1, X(1))$ *is a minimum is the matrix where*

$$\eta_i = \frac{\eta}{\delta}\,\delta_i \qquad i = 1, 2, \ldots, n,$$

where

$$\delta_i = \left[1 - \sum_{j \neq i} k_j\right](1 - k_i) \text{ and } \delta = \left[\prod_{i=1}^{n} \delta_i\right]^{1/n}.$$

Proof. It can be readily be shown that

$$s_{n-1} = \frac{\eta^n}{\prod_{j=1}^{n}(1-k_j)}\left[\sum_{j=1}^{n-1}\frac{\delta_j}{\eta_j} + \frac{\delta_n}{\eta^n}\prod_{j=1}^{n-1}\eta_j\right].$$

Then

$$\frac{\partial s_{n-1}}{\partial \eta_i} = \frac{\eta^n}{\prod_{j=1}^{n}(1-k_j)}\left[\frac{-\delta_i}{\eta_i^2} + \frac{\delta_n}{\eta^n}\prod_{\substack{j=1\\ j\neq 1}}^{n-1}\eta_j\right] = 0$$

$$\frac{\delta_i}{\eta_i^2} = \frac{\delta_n}{\eta^n}\prod_{\substack{j=1\\ j\neq 1}}^{n-1}\eta_j\frac{\eta_i\eta_n}{\eta_i\eta_n} = \frac{\delta_n}{\eta^n}\frac{\eta^n}{\eta_i\eta_n} = \frac{\delta_n}{\eta_i\eta_n}$$

or $\frac{\delta_i}{\eta_i} = \frac{\delta_n}{\eta_n}$ for $i = 1,2,\ldots,n-1$. This equation obviously holds for $i = n$ as well. Therefore $\prod_{i=1}^{n}\frac{\delta_i}{\eta_i} = \left[\frac{\delta_n}{\eta_n}\right]^n$ which implies $\frac{\delta}{\eta} = \frac{\delta_n}{\eta^n}$. Thus, $\frac{\delta_i}{\eta_i} = \frac{\delta}{\eta}$ or $\eta_i = \frac{\eta}{\delta}\delta_i$.

This completes the proof.

Denoting the matrix whose main diagonal elements are equal by A_e, the one whose characteristic polynomial has a minimum slope at $(1, X(1))$ by A_m and the matrix with smallest λ by A^*, it can readily be shown that in two dimensions

$$A_e = A_m = A^*.$$

However, in three dimensions, this is not the case.

Let

$$X_e = (x-1)^3 + s_1(x-1)^2 + s_2(x-1) + s_3$$

and

$$X_m = (x-1)^3 + s'_1(x-1)^2 + s'_2(x-1) + s'_3.$$

Since $s_3 = s'_3$, $s'_2 < s_2$ by definition of X_m and $s < s'_1$, the geometric means being less than the arithmetic means, we have

$$X_e - X_m = (x-1)[(s_1 - s'_1)(x-1) + (s_2 - s'_2)] < 0 \quad 0 < x < 1.$$

Thus $X_e < X_m$ for $0 < x < 1$ and so when $X_m(\lambda_m) = 0$, $X_e(\lambda_m) < 0$ implying that $\lambda_m < \lambda_e$. We see that in three dimensions, A_m is better than A_e. However, it is an open question whether it is the "best". For dimensions greater than three, we do not even know that much.

Fortunately, since $0 < k_i < 1/n$, all the k_i's are close to each other, especially for large n. But when all k's are equal, $A_e = A_m = A^*$ by Theorem 6. This means that we can use A_e or A_m as good approximations to the elusive A^*.

References

[1] Barsky, M. (1972), “Repeated convergence and fractional differences.”, Annales Polonici Mathematici,26.

[2] Barsky, M. (1970. “On extending the theory of Cesaro summability”, Annales Polonici Mathematicii, 23.

[3] Berman, A. And Plemmons, R. (1979). Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York.

[4] Davis, P. (1965). The Mathematics of Matrices, Blaisdell Publishing Company. Waltham, MA.

[5] Graham, A. (1987). Nonnegative Matrices and Applicable Topics in Linear Algebra, Ellis Howard Ltd and John Wiley & Sons, New York.

[6] Kmenta, J. (971). Elements of Econometrics, MacMillan Company, New York,NY.

[7] Levin R., Kirkpatrick, C.,and Rubin, D. (1982). Quantitative Approaches to Management (5th ed.), McGraw Hill, New York, NY.

[8] Mirsky, L., (1961). An Introduction to Linear Algebra, Oxford University Press, Amen House, London E.C.4.

[9] Schneider, H., and Baker, G., (1968). Matricies and Linear Algebra, Holt, Renehart, and Winston, Inc., New York, NY.

[10] Senata, E., (1980). Nonnegative Matrices and Markov Chains, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin.